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ON UNITARY PERFECT NUMBERS

by



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ABSTRACT

A divisor d of a positive integer n is called a unitary divisor of n if d and n/d are relatively prime. $\sigma^*(n)$ represents the sum of the unitary divisors of n . n is called unitary perfect if $\sigma^*(n) = 2n$. Let $N = 2^m p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}$ be unitary perfect. Known results concerning all possible values of N for certain fixed values of m and r have been verified independently and extended to prove that 6, 60, 90, and 87360 are the only unitary perfect numbers, N , such that $m \leq 11$ or $r \leq 6$.

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CHAPTER I INTRODUCTION

The Fundamental Theorem of Arithmetic states that every positive integer n greater than 1 can be uniquely represented as follows

$$(1.1) \quad n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}$$

where r and α_i , $1 \leq i \leq r$ are positive integers and each p_i , $1 \leq i \leq r$, is a prime number such that $p_1 < p_2 < \dots < p_r$. The right hand side of equation (1.1) is sometimes called the standard form (of the prime factorization) of n .

An arithmetic function is any complex-valued function that has as its domain the set of positive integers. The arithmetic functions of interest in number theory are those which are in a simple way dependent upon the standard form of n . An example of this type of function is $\sigma(n)$ which represents the sum of the divisors of n , i.e.,

$$\sigma(n) = \sum_{d|n} d.$$

Definition 1. Two positive integers n and m are relatively prime if their greatest common divisor is 1, i.e., $(m,n) = 1$.

Definition 2. An arithmetic function f is called multiplicative if $f(mn) = f(m)f(n)$ whenever $(m,n) = 1$.

It is easily seen that if the value of a multiplicative function is known for p^α for all primes p and all positive integers α , the value of the function is then uniquely defined for all positive integers n .

The following theorem is well known.

Theorem 1. The function σ is multiplicative and if

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}$$

then

$$\begin{aligned} \sigma(n) &= (1 + p_1 + p_1^2 + \dots + p_1^{\alpha_1}) \dots (1 + p_r + p_r^2 + \dots + p_r^{\alpha_r}) \\ &= \prod_{i=1}^r \frac{p_i^{\alpha_i+1} - 1}{p_i - 1}. \end{aligned}$$

We give a proof for completeness. Clearly

$$\sigma(p^\alpha) = 1 + p + p^2 + \dots + p^\alpha.$$

$$1 + p + p^2 + \dots + p^\alpha = \frac{p^{\alpha+1} - 1}{p - 1}$$

by a well known identity. Let $(m,n) = 1$. Then by the

Fundamental Theorem of Arithmetic the product of a divisor of m and a divisor of n is a divisor of mn , and every divisor of mn can be represented as the product of a divisor of m and a divisor of n . Therefore σ is multiplicative and if

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}$$

then

$$\sigma(n) = (1 + p_1 + p_1^2 + \dots + p_1^{\alpha_1}) \dots (1 + p_r + p_r^2 + \dots + p_r^{\alpha_r}) .$$

Definition 3. A divisor d of a positive integer n is called a unitary divisor of n if $(d, n/d) = 1$.

Definition 4. $\sigma^*(n)$ represents the sum of the unitary divisors of n , i.e.,

$$\sigma^*(n) = \sum_{\substack{d|n \\ (d, n/d)=1}} d .$$

Theorem 2. The function σ^* is multiplicative and if

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}$$

then

$$\sigma^*(n) = (1 + p_1^{\alpha_1}) \dots (1 + p_r^{\alpha_r}) .$$

Proof. σ^* is multiplicative by the same argument used to prove that σ is multiplicative. Clearly $\sigma^*(p^\alpha) = 1 + p^\alpha$. The theorem follows.

Definition 5. A positive integer n is called perfect if it is equal to the sum of its proper divisors, i.e. n is perfect if $\sigma(n) = 2n$.

It has been known since Euclid's time that if n is even and is of the form

$$(1.2) \quad n = 2^{p-1}(2^p - 1)$$

with both p and $2^p - 1$ prime then n is perfect. This can easily be proved using Theorem 1, since

$$\sigma(n) = \sigma(2^{p-1})\sigma(2^p - 1) = (2^p - 1)(2^p) = 2n.$$

Euler also proved the converse result that all even perfect numbers must be of form (1.2).

Primes of the form $2^p - 1$ where p is a prime are called Mersenne primes. As was stated above, the problem of finding even perfect numbers is equivalent to that of finding Mersenne primes. The only primes p for $p < 20000$ which give corresponding Mersenne primes are 2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107, 127, 521, 607, 1279, 2203, 2281, 3217, 4253, 4423, 9689, 9941,

11213, and 19937 . [6] It is not known if there are infinitely many of these primes and therefore it is not known if there are infinitely many even perfect numbers. It is not known if there are any odd perfect numbers.

Definition 6. A positive integer n is called unitary perfect if the sum of its unitary divisors is equal to $2n$.

It is trivial to prove that there are no odd unitary perfect numbers.

We note the following:

The only number that is both perfect and unitary perfect is 6 .

Proof. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}$ be both perfect and unitary perfect. Then

$$(1+p_1^{\alpha_1}) \dots (1+p_r^{\alpha_r}) = (1+p_1 + \dots + p_1^{\alpha_1-1})(1+p_r + \dots + p_r^{\alpha_r-1})$$

which implies that n is square-free. Next, n is even because there are no odd unitary perfect numbers. Thus, $n = 2^{p-1}(2^p - 1)$ because n is perfect. Since n is square-free, $p = 2$ which gives $n = 6$ as the only perfect and unitary perfect number.

Subbarao and Warren [5] have shown that if $2^m p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}$ is a unitary perfect number then 6, 60, 90, and 87360 are the only unitary perfect numbers for which $m \leq 7$ or $r \leq 5$. Subbarao [4] has further reported that in an unpublished paper by Cook et al. [1] it has been proved that the four unitary perfect numbers listed above are the only unitary perfect numbers such that $m \leq 10$ or $r \leq 6$. After studying a copy of this unpublished paper the author feels that the proofs in the paper seem to contain some errors. In Chapter III the results of Cook et al. have been verified independently and have been extended to show that the four numbers listed are the only unitary perfect numbers with $m \leq 11$ or $r \leq 6$. Also alternate proofs which are somewhat simpler than those given by Subbarao and Warren [5] are given for the result that 6, 60, 90, and 87360 are the only unitary perfect numbers such that $m \leq 7$ or $r \leq 5$. In Chapter II we prepare the background for the results of Chapter III.

Subbarao and Warren [5] have proved the following two theorems.

Theorem 3. Let m be a fixed positive integer.

There are at most a finite number of unitary perfect numbers N such that 2^m is a unitary divisor of N .

Theorem 4. There are at most a finite number of unitary perfect numbers with a fixed number of primes.

It has been conjectured by Subbarao and Warren [4,5] that there are only finitely many unitary perfect numbers.

Wall [8] has discovered a fifth unitary perfect number,

$$2^{18} \cdot 3 \cdot 5^4 \cdot 7 \cdot 11 \cdot 13 \cdot 19 \cdot 37 \cdot 79 \cdot 109 \cdot 157 \cdot 313$$

and has proved that this number is indeed the next unitary perfect number after 87360 . Wall [7] has conjectured that there are infinitely many unitary perfect numbers.

CHAPTER II PRELIMINARY RESULTS

Notation

In this chapter and the next N represents an even integer given by $N = 2^m n$ with m a positive integer and n an odd positive integer greater than 1. With each odd integer n there are associated three non-negative integers not all zero, represented by the letters a , b , and c . The letter a represents the number of distinct prime divisors, p , of n such that $p \equiv 1 \pmod{4}$. The letter b represents the number of distinct prime divisors, p , of n such that $p \equiv 3 \pmod{4}$ and p occurs with even exponent. The letter c represents the number of distinct prime divisors, p , of n such that $p \equiv 3 \pmod{4}$ and p occurs with odd exponent. $K(a,b,c)$ denotes the set of all odd numbers n associated as defined above with the integers a , b , and c . For a prime p , $p^\alpha || x$ means that p^α is a unitary divisor of x .

Some Lemmas

By definition if $N = 2^m n$ is unitary perfect then

$$(2.1) \quad (1 + 2^m)(1 + p_1^{\alpha_1}) \dots (1 + p_r^{\alpha_r}) = 2^{m+1} p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}.$$

Therefore,

$$(2.2) \quad \sigma^*(n) = 2^{m+1}/(1 + 2^m)$$

The following lemmas are used in the proofs of the results contained in Chapter III.

Lemma 1. $f(m) = 2^{m+1}/(1 + 2^m)$ is a monotonically increasing function and $\lim_{m \rightarrow \infty} f(m) = 2$.

The proof of Lemma 1 is trivial.

Lemmas 2 and 3 are results proved by Subbarao and Warren in [5].

Lemma 2. If $N = 2^m n$ is unitary perfect then

$$a + b + 2c \leq a + b + \sum_i iC_i = m + 1$$

where a , b , and c are as defined above and C_i is the number of distinct divisors, p , of n such that $p \equiv 3 \pmod{4}$, p occurs with an odd exponent, and 2^i is a unitary divisor of $(1 + p)$. If $c = 0$ then $a + b = m + 1$.

Proof. If $p \equiv 1 \pmod{4}$ then $p^\alpha \equiv 1 \pmod{4}$ for any positive integer α . Therefore, $1 + p^\alpha \equiv 2 \pmod{4}$ which implies that $2 \parallel (1 + p^\alpha)$. If $p \equiv 3 \pmod{4}$ then $p^\alpha \equiv 1 \pmod{4}$ for any even positive integer α . Thus, as before, $2 \parallel (1 + p^\alpha)$. If $p \equiv 3 \pmod{4}$ then $p^\alpha \equiv 3 \pmod{4}$ for any odd positive integer α . Thus, $4 \mid (1 + p^\alpha)$. In fact, $2^i \parallel (1 + p^\alpha)$ if and only if $2^i \parallel (1 + p)$. Thus, in the prime factorization of the left hand side of equation (2.1) the exponent of 2 is equal to

$$a + b + \sum_i i c_i$$

which is greater than or equal to $a + b + 2c$. The lemma follows.

Lemma 3. If $N = 2^m n$ is unitary perfect with

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}$$

and if 3 does not divide n then m and r are even integers.

Proof. Assume that m is odd. Then

$2^m + 1 \equiv 0 \pmod{3}$ which implies that $3 \mid n$ which is a contradiction. Therefore, m is even and $2^{m+1} \equiv 2 \pmod{3}$.

If $p_i^{\alpha_i} \mid n$ then $p_i^{\alpha_i} \equiv 1 \pmod{3}$. Therefore, $2^{m+1} n \equiv 2 \pmod{3}$ and $(1 + 2^m)^{\sigma^*(n)} \equiv 2^{r+1} \pmod{3}$.

Thus, from (2.1) $2^{r+1} \equiv 2 \pmod{3}$ which implies that r is even.

Lemma 4. Let $N = 2^m n$ be unitary perfect.

- (a) If 3, 5, 7, and 13 are unitary divisors of n then every other unitary divisor of n must be > 64 .
- (b) If 3, 5, 7, and 11 are unitary divisors of n then every other unitary divisor of n must be > 384 .
- (c) If 3 is a unitary divisor of n and if 5 does not divide n then $N' = 3 \cdot 5 \cdot 2^m n$ is also unitary perfect.
- (d) If 7 is a unitary divisor of n and if 5 and 13 do not divide n then $N' = 5^2 \cdot 7 \cdot 13 \cdot 2^m n$ is also unitary perfect.

(2.3) Proof of parts (a) and (b). Let $k = 3 \cdot 5 \cdot 7 \cdot 13 \cdot 64$.

Let $k' = 3 \cdot 5 \cdot 7 \cdot 11 \cdot 384$. Then $\sigma^*(k)/k = \sigma^*(k')/k' = 2$.

Therefore if there are any unitary divisors p^α of n such that $p^\alpha \leq 64$ in case (a) and $p^\alpha \leq 384$ in case (b), $\sigma^*(n)/n \geq 2$. However, by (2.2) and Lemma 1, $\sigma^*(n)/n < 2$ for any fixed m such that $N = 2^m n$ is unitary perfect.

Proof of part (c). Let $n = 3k$. Then

$$N' = 3^2 \cdot 5 \cdot 2^m k. \quad \sigma^*(n)/n = \frac{4}{3} \frac{\sigma^*(k)}{k} = 2^{m+1}/(1 + 2^m), \text{ and}$$

$$\sigma^*(3^2 \cdot 5k)/(3^2 \cdot 5k) = \frac{4}{3} \frac{\sigma^*(k)}{k} = 2^{m+1}/(1 + 2^m). \text{ Therefore}$$

N' is unitary perfect.

Proof of part (d). Let $j = 5^2 \cdot 7^2 \cdot 13$. Observe

that $\sigma^*(j)/j = 8/7$. The proof is similar to that for part (c).

An example of two unitary perfect numbers N and N' satisfying the conditions of Lemma 4, part (c) are $N = 6$ and $N' = 90$. There are no known unitary perfect numbers satisfying the conditions of part (d) of the lemma.

The following three lemmas are useful in comparing the values of $\sigma^*(n)/n$ for various values of n .

Lemma 5. If

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}$$

and

$$n' = q_1^{\beta_1} q_2^{\beta_2} \dots q_s^{\beta_s}$$

with $r \geq s$ then $\sigma^*(n)/n \geq \sigma^*(n')/n'$ whenever

$$q_1^{\beta_1} \geq p_1^{\alpha_1}, q_2^{\beta_2} \geq p_2^{\alpha_2}, \dots, q_s^{\beta_s} \geq p_s^{\alpha_s}.$$

The proof of Lemma 5 is trivial.

Lemma 6. Let $n = p_1 p_2 \dots p_r$ with each p_i a prime. If it is required to square k of the prime factors of n to obtain another integer, say n' , $k \leq r$, then the largest value of $\sigma^*(n')/n'$ is obtained when the largest k prime factors of the r prime factors of n are squared.

Proof. Let $r = 2$, $k = 1$. Then $n = p_1 p_2$.

$$\frac{(p_1^2 + 1)(p_2 + 1)}{p_1^2 p_2} < \frac{(p_1 + 1)(p_2^2 + 1)}{p_1 p_2^2}$$

$$\leftrightarrow (p_1^2 + 1)(p_2^2 + p_2) < (p_1^2 + p_1)(p_2^2 + 1)$$

$$(*) \leftrightarrow p_1^2 p_2 + p_2^2 + p_2 < p_1 p_2^2 + p_1^2 + p_1$$

Let $p_2 = m + p_1$. (*) becomes

$$p_1^2(m + p_1) + (m + p_1)^2 + m + p_1 < p_1(m + p_1)^2 + p_1^2 + p_1$$

$$(**) \leftrightarrow (p_1 - 1)^2 + m(p_1 - 1) > 2$$

(**) is valid since $p \geq 2$ and $m \geq 2$. Therefore,

$$\frac{(p_1^2 + 1)(p_2 + 1)}{p_1^2 p_2} < \frac{(p_1 + 1)(p_2^2 + 1)}{p_1 p_2^2}$$

The lemma follows easily.

Lemma 7. (a) Let two positive integers, M and M' , have prime factorizations which contain the same number of distinct prime factors and such that the unitary prime power divisors of M are identical to the unitary prime power divisors of M' except for two, $p_s^{\alpha_s}$ and $p_t^{\alpha_t}$

for M and $q_u^{\beta_u}$ and $q_v^{\beta_v}$ for M' , such that

$$p_s^{\alpha_s} < q_u^{\beta_u} < q_v^{\beta_v} < p_t^{\alpha_t}. \text{ Let } x = p_s^{\alpha_s}, y = p_t^{\alpha_t},$$

$$x' = q_u^{\beta_u}, y' = q_v^{\beta_v}, k = y - x, m = x' - x, n = y - y',$$

$$L = n - m, J = k - n.$$

(i) If $m \geq n$ then $\sigma^*(M)/M > \sigma^*(M')/M'$.

(ii) If $m < n$ then $\sigma^*(M)/M > \sigma^*(M')/M'$ if

and only if

$$(*) \quad mJ(2x + n + J + 1) > Lx(x + 1) .$$

In addition, if $m < n$ and $mJ \geq Lx$ or $mJ \geq x(x + 1)$

then $\sigma^*(M)/M > \sigma^*(M')/M'$.

(b) If the prime factorizations of two positive integers, K and K' , contain the same number of distinct prime fac-

tors and if the unitary prime power divisors of K are

identical to the unitary prime power divisors of K'

except for two, $p_s^{\alpha_s}$ and $p_t^{\alpha_t}$ for K and $q_u^{\beta_u}$ and

$q_v^{\beta_v}$ for K' such that $p_s^{\alpha_s} < q_u^{\beta_u} < q_v^{\beta_v} < p_t^{\alpha_t}$,

then $\sigma^*(K)/K > \sigma^*(K')/K'$ whenever there exist four other prime powers, $p_s^{\alpha_s}$, $p_t^{\alpha_t}$, $q_u^{\beta_u}$, and $q_v^{\beta_v}$, such that $p_t^{\alpha_t} = p_t^{\alpha_t}$,

$q_v^{\beta_v} = q_v^{\beta_v}$, $p_s^{\alpha_s} \leq p_s^{\alpha_s}$ and $q_u^{\beta_u} \geq q_u^{\beta_u}$, and

$$\frac{(p_s^{\alpha_s} + 1)(p_t^{\alpha_t} + 1)}{p_s^{\alpha_s} p_t^{\alpha_t}} > \frac{(q_u^{\beta_u} + 1)(q_v^{\beta_v} + 1)}{q_u^{\beta_u} q_v^{\beta_v}}$$

Proof of part (a).

$$\frac{(x + 1)(y + 1)}{xy} > \frac{(x' + 1)(y' + 1)}{x'y'}$$

$$\Leftrightarrow \frac{(x + 1)(x + k + 1)}{x(x + k)} > \frac{(x + m + 1)(x + k - n + 1)}{(x + m)(x + k - n)}$$

$$\begin{aligned} \Leftrightarrow & (x + 1)(x + k + 1)(x + m)(x + k - n) \\ & > x(x + k)(x + m + 1)(x + k - n + 1) \end{aligned}$$

$$(**) \Leftrightarrow mx^2 + 2mkx + mx + mk^2 + mk > nx^2 + 2mnx + nx + mnk + mn$$

$k > n$ by definition. Therefore $(**)$ holds if $m \geq n$. If $m < n$ let $L = n - m$ and $J = k - n$. Then $(**)$ becomes

$$mx^2 + 2m(m + L + J)x + mx + m(m + L + J)^2 + m(m + L + J) \\ > (m + L)x^2 + 2m(m + L)x + (m + L)x + (m + L)mk + m(m + L)$$

which holds if and only if $(*)$ holds. $mJ \geq Lx$ implies

$(*)$ and $mJ \geq x(x + 1)$ implies $(*)$. Therefore part (a)

of the lemma is proved. The proof of part (b) is trivial.

An example of the usefulness of Lemma 7 is the following taken from page 62.

$$n_7 = 5 \cdot 13 \cdot 17 \cdot 3 \cdot 7 \cdot 11 \cdot 19 \cdot 683^2$$

$$n_8 = 5 \cdot 13 \cdot 17 \cdot 29 \cdot 3 \cdot 7 \cdot 11 \cdot 683$$

It is determined that $\sigma^*(n_8)/n_8 < \sigma^*(n_7)/n_7$ by use of Lemma 7

part (a) with $p_s^{\alpha_s} = 19$, $p_t^{\alpha_t} = 683^2$, $q_u^{\beta_u} = 29$, $q_v^{\beta_v} = 683$.

$$n_{14} = 5 \cdot 13 \cdot 17 \cdot 29 \cdot 37 \cdot 3 \cdot 7 \cdot 11 \cdot 683^2$$

$$n_{15} = 5 \cdot 13 \cdot 17 \cdot 29 \cdot 37 \cdot 41 \cdot 3 \cdot 7 \cdot 683$$

$\sigma^*(n_{15})/n_{15} < \sigma^*(n_{14})/n_{14}$ by Lemma 7 part (b) with $p_s^{\alpha_s} = 11$,

$p_t^{\alpha_t} = 683^2$, $q_u^{\beta_u} = 41$, $q_v^{\beta_v} = 683$, and $p_s^{\alpha_s}$, $p_t^{\alpha_t}$,

$q_u^{\beta_u}$, $q_v^{\beta_v}$ defined as above for n_7 and n_8 .

CHAPTER III. EXAMINATION OF SOME POSSIBLE VALUES OF m

AND r FOR THE UNITARY PERFECT NUMBER

$$N = \frac{2^m p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}}{p_1 p_2 \dots p_r}$$

Statement of Results to be Proved

Subbarao and Warren [5] proved parts (1) through (8) of the following theorem. Cook et al, [1] attempted to add parts (9) and (10). As observed in the introduction the proofs used by Cook et al. seem to contain some errors. Therefore parts (9) and (10) of the theorem have been verified independently.

Theorem 1. Let $N = \frac{2^m p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}}{p_1 p_2 \dots p_r}$ be unitary perfect.

- (1) If $r = 1$, then $N = 6$.
- (2) If $m = 1$, then $N = 6$ or 90 .
- (3) If $m = 2$, then $N = 60$.
- (4) If $r = 2$, then $N = 60$ or 90 .
- (5) It is not possible for $m = 3, 4, 5$, or 7 .
- (6) It is not possible for $r = 3$ or 5 .
- (7) If $m = 6$, then $N = 87360$.
- (8) If $r = 4$, then $N = 87360$.
- (9) It is not possible for $m = 8, 9$, or 10 .
- (10) It is not possible for $r = 6$.

Theorem 1 has been extended in this work to obtain the following theorem.

Theorem 2. Let $N = 2^m n$ be unitary perfect.

It is not possible for $m = 11$.

General Discussion of Methods of Proof Used

For convenience in the calculations in the following proofs Tables I and II have been prepared. Table I contains all primes p such that $p < 200$ and $p \equiv 1 \pmod{4}$. It also shows for which of these primes p , $(1 + p)$ is divisible by 3. Table II gives the same information for primes $p \equiv 3 \pmod{4}$, $p < 420$. Table II also gives the maximum power of 2 which divides $(1 + p)$.

Definition 1. For any two positive integers n and m , $m <^* n$ or $n >^* m$ will be used in the following to denote the fact that $\sigma^*(m)/m < \sigma^*(n)/n$.

When determining if there are any unitary perfect numbers with $m = m'$ the first step is the following.

Step 1. From Lemma 2, $a + b + 2c \leq m' + 1$.

Therefore all possible sets (a, b, c) such that

$a + b + 2c = m' + 1$ are determined. Each of these sets

(a, b, c) represents all sets $K(a', b', c')$ such that

$a' \leq a$, $b' \leq b$, and $c' \leq c$, and is called a class.

Thus, every possible value of n such that $N = 2^{m'}$ n

is unitary perfect is represented in these classes.

A set (a', b', c') such that $a' \leq a$, $b' \leq b$, and $c' \leq c$,

not all equal, is called a subclass of class (a, b, c) .

Table I. Information on Divisibility by 3 of $(1 + p)$
for all Primes p such that $p < 200$ and $p \equiv 1 \pmod{4}$

Prime	$(1 + p) \equiv 0 \pmod{3} ?$
5	yes
13	no
17	yes
29	yes
37	no
41	yes
53	yes
61	no
73	no
89	yes
97	no
101	yes
109	no
113	yes
137	yes
149	yes
157	no
173	yes
181	no
193	no
197	yes

Table II. Some Properties of all Primes p such that $p < 420$ and $p \equiv 3 \pmod{4}$

Prime	$(1 + p) \equiv 0 \pmod{3} ?$	$i \ni 2^i \parallel (1 + p)$
3	no	2
7	no	3
11	yes	2
19	no	2
23	yes	3
31	no	5
43	no	2
47	yes	4
59	yes	2
67	no	2
71	yes	3
79	no	4
83	yes	2
103	no	3
107	yes	2
127	no	7
131	yes	2
139	no	2
151	no	3
163	no	2

Table II. Continued

Prime	$(1 + p) \equiv 0 \pmod{3} ?$	$i \ni 2^i \parallel (1 + p)$
167	yes	3
179	yes	2
191	yes	6
199	no	3
211	no	2
223	no	5
227	yes	2
239	yes	4
251	yes	2
263	yes	3
271	no	4
283	no	2
307	no	2
311	yes	3
331	no	2
347	yes	2
359	yes	3
367	no	4
379	no	2
383	yes	7
419	yes	2

From (2.1) it is known that $(1 + 2^{m'})|n$.

Therefore $(1 + 2^{m'})$ is calculated, and depending on its prime factorization, some of the above classes may be eliminated from consideration. For example if 5 divides $(1 + 2^{m'})$ then all classes with $a = 0$ may be eliminated. If m' is odd then $3|n$ by Lemma 3 and therefore all classes such that $b + c = 0$ can also be eliminated.

Using Tables I and II, Lemmas 5 and 6, and the fact that $(1 + 2^{m'})|n$, the number n is determined such that n gives the maximum value of $\sigma^*(n)/n$ for all possible n in each of the remaining classes. This number is listed next to its corresponding class. The classes are numbered and n_j refers to the number n that has been determined to give the maximum value of $\sigma^*(n)/n$ under the above conditions for class j .

Using Lemmas 5 and 7 the direction of the $<^*$ relationship of Definition 1 of this chapter is determined for various pairs of the n_j 's. In the following proofs the $<^*$ symbol which is on the same line as and immediately precedes a number, n_j , refers to the $<^*$ relationship between n_j and n_{j-1} .

By direct computation and by use of (2.3) it is then determined which of the n_j satisfy the following.

$$(3.1) \quad \sigma^*(n_j)/n_j \leq 2^{m'+1}/(1 + 2^{m'})$$

Assume equality holds in (3.1) for n_j . By (2.2) $N = 2^{m'} n_j$ is unitary perfect. Since n_j is the number n that gives the maximum value of $\sigma^*(n)/n$ for all possible n in class j there are no other numbers n in class j such that $N = 2^{m'} n$ is unitary perfect. Assume (3.1) holds with the inequality sign for n_j . Then $N = 2^{m'} n_j$ is not unitary perfect and as before there will be no numbers, n , in class j such that $N = 2^{m'} n$ is unitary perfect.

Therefore only the classes j such that $\sigma^*(n_j)/n_j > 2^{m'+1}/(1 + 2^{m'})$, if any, are left to be considered in determining any unitary perfect numbers $N = 2^{m'} n$.

Step 2 varies depending on the particular proof under consideration. However, in many cases the following is done next. For class j , say, n_j is modified by assuming that for all n in class j , $3 \nmid n$. The resulting number is called $n_{j,3}$. n_j is modified again by assuming that for all n in class j , $5 \nmid n$ to obtain a new number, $n_{j,5}$. Again the direction of the $<^*$ relationship between various pairs of the $n_{j,3}$'s and the $n_{j,5}$'s is determined. It can then be determined for which of the n_j 's, $3 \parallel n_j$ and/or $5 \parallel n_j$. If $3 \mid (1 + 2^{m'})$ consider the classes for which $3 \parallel n_j$ and $5 \parallel n_j$. 3^2 divides the left hand side of

equation (2.1) which is a contradiction since 3 is a unitary divisor of the right hand side of the equation. Therefore, these classes can be eliminated from consideration.

Other methods that are used are considered individually in the following proofs.

When determining if there are any unitary perfect numbers $N = 2^m n$ with $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}$ and $r = r'$ the above method is modified.

Step 1 is to determine all possible sets (a,b,c) such that $a + b + c = r'$. As before, each set (a,b,c) is called a class. However, in these parts of the proof, a class represents only those numbers in the set $K(a,b,c)$ and not those numbers in the sets $K(a',b',c') \neq K(a,b,c)$, $a' \leq a$, $b' \leq b$, and $c' \leq c$.

By Lemma 2 if $c = 0$ then $a + b = m + 1$.

For the proofs of these parts of the theorem $(a + b)$ will always be less than or equal to 6. Any previously proved parts of the theorem may be used and therefore all classes with $c = 0$ will have already been investigated.

As before, using Tables I and II and Lemmas 5 and 6 the number n giving the maximum value of $\sigma^*(n)/n$ is determined for each of the remaining classes. The numbers are listed on the same line as their corresponding class, the classes are numbered, and n_j is used to refer to

the number corresponding to class j .

Using Lemmas 5 and 7 the direction of the $\prec^*$ relationship is then determined between various pairs of the n_j . The $\prec^*$ symbol on the same line as and immediately preceding n_j refers to the $\prec^*$ relationship between n_j and n_{j-1} .

By direct computation and by use of (2.3) it is then determined which of the n_j satisfy (3.2).

$$(3.2) \quad \sigma^*(n_j)/n_j \leq 2^{k+1}/(1 + 2^k)$$

where k is any integer less than or equal to one plus the maximum value, m' , for which the existence of unitary perfect numbers with $m = m'$ has been investigated.

Assume equality holds in (3.2) for n_j . By (2.2) $N = 2^k n_j$ is unitary perfect. Since n_j gives the maximum value of $\sigma^*(n)/n$ for all n in class j , if there are any other numbers n in class j such that

$$(3.3) \quad \sigma^*(n)/n = 2^{k+1}/(1 + 2^k)$$

k defined as above, they will have been determined in previous proofs. Similarly if inequality holds in (3.2) for n_j , any numbers n in class j for which (3.3) holds will have been determined in previous proofs. Therefore the classes for which (3.2) holds can then be

eliminated from consideration.

Step 2. For those classes j , if any, for which $\sigma^*(n_j)/n_j > 2^{k+1}/(1 + 2^k)$ for k equal to one plus the maximum value, m' , for which the existence of unitary perfect numbers with $m = m'$ has been investigated, $n_{j,3}$ and $n_{j,5}$ are determined as described on page 23. The method of proof then varies depending on which part of the theorem is being proved. In most instances, however, it is then determined for which values of j , $3 \parallel n_j$ and $5 \parallel n_j$. Using Tables I and II, n_j for these classes is then modified taking into consideration the fact that since $3 \parallel n_j$ and $5 \parallel n_j$ no other prime power, p^α , such that $3|(1 + p^\alpha)$ is a unitary divisor of n_j . Then, for these modified values of n_j it is determined for which values of j , (3.2) is satisfied. This may eliminate more classes.

For those classes which still remain to be considered different methods are used which are discussed individually in the following proofs.

In the following, unless otherwise specified, n represents the numbers that satisfy the equation $N = 2^m n$ with the restrictions implied by the particular proof under consideration. At any point in the following proofs, the number, n , in class j giving the maximum value of $\sigma^*(n)/n$ for all n , defined as above, in class j is

represented by n_j . k_j represents a similar value for all numbers in subclass j . M_j represents a similar value for all numbers in $K(a,b,c)$ where a , b , and c are the three numbers associated with class j . $n_{j,i}$ represents the number that is determined in the same manner as n_j with the additional assumption that $i \nmid n$ for all n in class j .

Individual Proofs for Various Values of m and r

Proof of Theorem 1 part (1). If $r = 1$, then

$N = 6$.

Step 1.

r is odd which implies that $3 \mid n$. Therefore $b + c \neq 0$.

Class	a	b	c	n_j
1	0	0	1	3
2	0	1	0	$> 3^2$

$\sigma^*(n_1)/n_1 = 2^2/3$. Therefore $N = 2 \cdot 3^2$ is unitary perfect with $m = 1$. By Lemma 5 any other number n in class 1 or 2 will give a value of $\sigma^*(n)/n$ which is less than $2^2/3$ and therefore by Lemma 1 not equal to $2^{m+1}/(1 + 2^m)$ for any positive integer m . Thus, 6 is the only unitary perfect number with $r = 1$.

Proof of Theorem 1 part (2). If $m = 1$, then

$N = 6$ or 90 .

Step 1.

$1 + 2 = 3$ which implies that $b + c \neq 0$.

Class	a	b	c	n_j
1	0	0	1	3
2	0	2	0	$3^2 \cdot 7^2$
3	1	1	0	$< * 5 \cdot 3^2$

For class 1, $r = 1$ and therefore $N = 6$ is the only unitary perfect number with n in class 1. $\sigma^*(n_3)/n_3 = 2^2/3$.
Therefore $N = 2 \cdot 3^2 \cdot 5 = 90$ is the only other unitary perfect number with $m = 1$.

Proof of Theorem 1 part (3). If $m = 2$, then

$N = 60$.

Step 1.

$1 + 2^2 = 5$ which implies that $a \neq 0$.

Class	a	b	c	n_j
1	1	0	1	$5 \cdot 3$
2	1	2	0	$5 \cdot 3^2 \cdot 7^2$
3	2	1	0	$< * 5 \cdot 13 \cdot 3^2$
4	3	0	0	$* > 5 \cdot 13 \cdot 17$

$\sigma^*(n_3)/n_3 < 2^3/5$. Therefore there are no unitary perfect numbers with $m = 2$ and n in classes 2, 3, or 4.

$\sigma^*(n_1)/n_1 = 2^3/5$. Therefore $N = 2^2 \cdot 3 \cdot 5 = 60$ is the only unitary perfect number with $m = 2$ and with n in class 1.

Proof of Theorem 1 part (4). If $r = 2$, then

$N = 60$ or 90 .

Step 1.

If $c = 0$ then $m = 1$. Therefore $N = 90$ is the only unitary perfect number with $r = 2$ such that $c = 0$.

Class	a	b	c	n_j
1	0	0	2	$3 \cdot 7$
2	0	1	1	$* > 3 \cdot 7^2$
3	1	0	1	$< * 5 \cdot 3$

As shown above $N = 60$ is the only unitary perfect number such that n is in class 3. $n_1 < * n_3$. Therefore any other number n in class 1, 2, or 3 will give a value of $\sigma^*(n)/n$ which is less than $2^3/5$. Therefore $m = 1$ for any other possible unitary perfect numbers with $r = 2$. Since all unitary perfect numbers with $m = 1$ are known, 60 and 90 are the only unitary perfect numbers with $r = 2$.

Proof of Theorem 1 part (5). It is not possible

for $m = 3$.

Step 1.

$1 + 2^3 = 3^2$ which implies that $b + c \neq 0$.

Class	a	b	c	n_j
1	0	0	2	$3^3 \cdot 7$
2	0	2	1	$7 \cdot 3^2 \cdot 11^2$
3	1	1	1	$<* 5 \cdot 7 \cdot 3^2$
4	2	0	1	$* > 5 \cdot 13 \cdot 3^3$
5	0	4	0	$3^2 \cdot 7^2 \cdot 11^2 \cdot 19^2$
6	1	3	0	$<* 5 \cdot 3^2 \cdot 7^2 \cdot 11^2$
7	2	2	0	$<* 5 \cdot 13 \cdot 3^2 \cdot 7^2$
8	3	1	0	$<* 5 \cdot 13 \cdot 17 \cdot 3^2$

$\sigma^*(n_j)/n_j < 2^4/9$ for $j = 1, 3$, and 8 . Therefore there are no unitary perfect numbers with $m = 3$.

Proof of Theorem 1 part (5). It is not possible for $m = 4$.

Step 1.

$1 + 2^4 = 17$ which implies that $a \neq 0$.

Class	a	b	c	n_j
1	1	0	2	$17 \cdot 3 \cdot 7$
2	1	2	1	$17 \cdot 3 \cdot 7^2 \cdot 11^2$
3	2	1	1	$<* 17 \cdot 5 \cdot 3 \cdot 7^2$
4	3	0	1	$<* 17 \cdot 5 \cdot 13 \cdot 3$
5	1	4	0	$17 \cdot 3^2 \cdot 7^2 \cdot 11^2 \cdot 19^2$
6	2	3	0	$<* 17 \cdot 5 \cdot 3^2 \cdot 7^2 \cdot 11^2$

Class	a	b	c	n_j
7	3	2	0	$<* 17 \cdot 5 \cdot 13 \cdot 3^2 \cdot 7^2$
8	4	1	0	$<* 17 \cdot 5 \cdot 13 \cdot 29 \cdot 3^2$
9	5	0	0	$* > 17 \cdot 5 \cdot 13 \cdot 29 \cdot 37$

$\sigma^*(n_j)/n_j < 2^5/17$ for $j = 1, 4$, and 8 . Therefore there are no unitary perfect numbers with $m = 4$.

Proof of Theorem 1 part (5). It is not possible for $m = 5$.

Step 1.

$1 + 2^5 = 3 \cdot 11$ which implies that $b + c \geq 2$.

Class	a	b	c	n_j
1	0	0	3	$3 \cdot 7 \cdot 11$
2	0	2	2	$3 \cdot 7 \cdot 11^2 \cdot 19^2$
3	1	1	2	$<* 5 \cdot 3 \cdot 7 \cdot 11^2$
4	2	0	2	$<* 5 \cdot 13 \cdot 3 \cdot 11$
5	0	4	1	$3 \cdot 7^2 \cdot 11^2 \cdot 19^2 \cdot 23^2$
6	1	3	1	$<* 5 \cdot 3 \cdot 7^2 \cdot 11^2 \cdot 19^2$
7	2	2	1	$<* 5 \cdot 13 \cdot 3 \cdot 7^2 \cdot 11^2$
8	3	1	1	$<* 5 \cdot 13 \cdot 17 \cdot 3 \cdot 11^2$
9	0	6	0	$3^2 \cdot 7^2 \cdot 11^2 \cdot 19^2 \cdot 23^2 \cdot 31^2$
10	1	5	0	$<* 5 \cdot 3^2 \cdot 7^2 \cdot 11^2 \cdot 19^2 \cdot 23^2$
11	2	4	0	$<* 5 \cdot 13 \cdot 3^2 \cdot 7^2 \cdot 11^2 \cdot 19^2$

Class	a	b	c	n_j
12	3	3	0	$<* 5 \cdot 13 \cdot 17 \cdot 3^2 \cdot 7^2 \cdot 11^2$
13	4	2	0	$<* 5 \cdot 13 \cdot 17 \cdot 29 \cdot 3^2 \cdot 11^2$

$\sigma^*(n_j)/n_j < 2^6/33$ for $j = 1, 4, 8$, and 13 . Therefore there are no unitary perfect numbers with $m = 5$.

Proof of Theorem 1 part (5). It is not possible for $m = 7$.

Step 1.

$1 + 2^7 = 3 \cdot 43$ which implies that $b + c \geq 2$.

Class	a	b	c	n_j
1	0	0	4	$3 \cdot 7 \cdot 11 \cdot 43$
2	0	2	3	$3 \cdot 7 \cdot 11 \cdot 19^2 \cdot 43^2$
3	1	1	3	$<* 5 \cdot 3 \cdot 7 \cdot 11 \cdot 43^2$
4	2	0	3	$<* 5 \cdot 13 \cdot 3 \cdot 7 \cdot 43$
5	0	4	2	$3 \cdot 7 \cdot 11^2 \cdot 19^2 \cdot 23^2 \cdot 43^2$
6	1	3	2	$<* 5 \cdot 3 \cdot 7 \cdot 11^2 \cdot 19^2 \cdot 43^2$
7	2	2	2	$<* 5 \cdot 13 \cdot 3 \cdot 7 \cdot 11^2 \cdot 43^2$
8	3	1	2	$<* 5 \cdot 13 \cdot 17 \cdot 3 \cdot 7 \cdot 43^2$
9	4	0	2	$* > 5 \cdot 13 \cdot 17 \cdot 29 \cdot 3 \cdot 43$
10	0	6	1	$3 \cdot 7^2 \cdot 11^2 \cdot 19^2 \cdot 23^2 \cdot 31^2 \cdot 43^2$
11	1	5	1	$<* 5 \cdot 3 \cdot 7^2 \cdot 11^2 \cdot 19^2 \cdot 23^2 \cdot 43^2$
12	2	4	1	$<* 5 \cdot 13 \cdot 3 \cdot 7^2 \cdot 11^2 \cdot 19^2 \cdot 43^2$

Class	a	b	c	n_j
13	3	3	1	$<* 5 \cdot 13 \cdot 17 \cdot 3 \cdot 7^2 \cdot 11^2 \cdot 43^2$
14	4	2	1	$<* 5 \cdot 13 \cdot 17 \cdot 29 \cdot 3 \cdot 7^2 \cdot 43^2$
15	5	1	1	$<* 5 \cdot 13 \cdot 17 \cdot 29 \cdot 37 \cdot 3 \cdot 43^2$
16	0	8	0	$3^2 \cdot 7^2 \cdot 11^2 \cdot 19^2 \cdot 23^2 \cdot 31^2 \cdot 43^2 \cdot 47^2$
17	1	7	0	$<* 5 \cdot 3^2 \cdot 7^2 \cdot 11^2 \cdot 19^2 \cdot 23^2 \cdot 31^2 \cdot 43^2$
18	2	6	0	$<* 5 \cdot 13 \cdot 3^2 \cdot 7^2 \cdot 11^2 \cdot 19^2 \cdot 23^2 \cdot 43^2$
19	3	5	0	$<* 5 \cdot 13 \cdot 17 \cdot 3^2 \cdot 7^2 \cdot 11^2 \cdot 19^2 \cdot 43^2$
20	4	4	0	$<* 5 \cdot 13 \cdot 17 \cdot 29 \cdot 3^2 \cdot 7^2 \cdot 11^2 \cdot 43^2$
21	5	3	0	$<* 5 \cdot 13 \cdot 17 \cdot 29 \cdot 37 \cdot 3^2 \cdot 7^2 \cdot 43^2$
22	6	2	0	$<* 5 \cdot 13 \cdot 17 \cdot 29 \cdot 37 \cdot 41 \cdot 3^2 \cdot 43^2$

$\sigma^*(n_j)/n_j < 2^8/129$ for $j = 1, 2, 6, 9, 15$, and 22 , and
 $\sigma^*(n_j)/n_j > 2^8/129$ for $j = 3$ and 7 . Therefore all classes
except $3, 4, 7$, and 8 can be eliminated from consideration.

Step 2.

$$\begin{aligned}
n_{3,3} &= 5 \cdot 7 \cdot 11 \cdot 43 \cdot 3^2 \\
*> n_{4,3} &= 5 \cdot 13 \cdot 3^3 \cdot 7 \cdot 43 \\
n_{7,3} &= 5 \cdot 13 \cdot 7 \cdot 11 \cdot 3^2 \cdot 43^2 \\
*> n_{8,3} &= 5 \cdot 13 \cdot 17 \cdot 7 \cdot 43 \cdot 3^2 \\
n_{3,5} &= 13 \cdot 3 \cdot 7 \cdot 11 \cdot 43^2 \\
*> n_{4,5} &= 13 \cdot 17 \cdot 3 \cdot 7 \cdot 43 \\
n_{7,5} &= 13 \cdot 17 \cdot 3 \cdot 7 \cdot 11^2 \cdot 43^2 \\
<*> n_{8,5} &= 13 \cdot 17 \cdot 5^2 \cdot 3 \cdot 7 \cdot 43^2
\end{aligned}$$

$\sigma^*(n_{j,i})/n_{j,i} < 2^8/129$ for $j,i = (3,3), (7,3), (3,5)$,
and $(8,5)$. Therefore $3 \nparallel n_j$ and $5 \nparallel n_j$ for the remaining
classes j which is a contradiction since $3 \mid (1 + 2^7)$.

Therefore there are no unitary perfect numbers with $m = 7$.

Proof of Theorem 1 part (6). It is not possible
for $r = 3$.

Step 1.

If $c = 0$ then $m = 2$. Therefore there are no unitary
perfect numbers with $r = 3$ such that $c = 0$.

Class	a	b	c	n_j
1	0	0	3	$3 \cdot 7 \cdot 11$
2	0	1	2	$* > 3 \cdot 7 \cdot 11^2$
3	1	0	2	$5 \cdot 3 \cdot 7$
4	0	2	1	$* > 3 \cdot 7^2 \cdot 11^2$
5	1	1	1	$5 \cdot 3 \cdot 7^2$
6	2	0	1	$< * 5 \cdot 13 \cdot 3$

$n_1 < * n_3 * > n_6$. $\sigma^*(n_3)/n_3 < 2^6/33$. Therefore there
are no unitary perfect numbers with $r = 3$.

Proof of Theorem 1 part (6). It is not possible
for $r = 5$.

Step 1.

If $c = 0$ then $m = 4$. Therefore there are no unitary
 perfect numbers with $r = 5$ such that $c = 0$.

Class	a	b	c	n_j
1	0	0	5	$3 \cdot 7 \cdot 11 \cdot 19 \cdot 23$
2	0	1	4	$* > 3 \cdot 7 \cdot 11 \cdot 19 \cdot 23^2$
3	1	0	4	$5 \cdot 3 \cdot 7 \cdot 11 \cdot 19$
4	0	2	3	$* > 3 \cdot 7 \cdot 11 \cdot 19^2 \cdot 23^2$
5	1	1	3	$5 \cdot 3 \cdot 7 \cdot 11 \cdot 19^2$
6	2	0	3	$< * 5 \cdot 13 \cdot 3 \cdot 7 \cdot 11$
7	0	3	2	$3 \cdot 7 \cdot 11^2 \cdot 19^2 \cdot 23^2$
8	1	2	2	$< * 5 \cdot 3 \cdot 7 \cdot 11^2 \cdot 19^2$
9	2	1	2	$< * 5 \cdot 13 \cdot 3 \cdot 7 \cdot 11^2$
10	3	0	2	$< * 5 \cdot 13 \cdot 17 \cdot 3 \cdot 7$
11	0	4	1	$3 \cdot 7^2 \cdot 11^2 \cdot 19^2 \cdot 23^2$
12	1	3	1	$< * 5 \cdot 3 \cdot 7^2 \cdot 11^2 \cdot 19^2$
13	2	2	1	$< * 5 \cdot 13 \cdot 3 \cdot 7^2 \cdot 11^2$
14	3	1	1	$< * 5 \cdot 13 \cdot 17 \cdot 3 \cdot 7^2$
15	4	0	1	$< * 5 \cdot 13 \cdot 17 \cdot 29 \cdot 3$

$n_4 < * n_1$. $\sigma^*(n_j)/n_j < 2^9/257$ for $j = 1, 9$, and 15 .

By (2.3) $\sigma^*(n_j)/n_j > 2$ for $j = 3, 5$, and 10 . Therefore

all classes except 3, 5, 6, and 10 can be eliminated from consideration.

Step 2.

Note that $3 \mid n$ since r is odd.

$$\begin{aligned}
 n_{3,3} &= 5 \cdot 3^3 \cdot 7 \cdot 11 \cdot 19 \\
 < * n_{5,3} &= 5 \cdot 7 \cdot 11 \cdot 19 \cdot 3^2 \\
 * > n_{6,3} &= 5 \cdot 13 \cdot 3^3 \cdot 7 \cdot 11 \\
 * > n_{10,3} &= 5 \cdot 13 \cdot 17 \cdot 3^3 \cdot 7 \\
 n_{3,5} &= 13 \cdot 3 \cdot 7 \cdot 11 \cdot 19 \\
 * > n_{5,5} &= 13 \cdot 3 \cdot 7 \cdot 11 \cdot 19^2 \\
 n_{6,5} &= 13 \cdot 17 \cdot 3 \cdot 7 \cdot 11 \\
 * > n_{10,5} &= 13 \cdot 17 \cdot 5^2 \cdot 3 \cdot 7
 \end{aligned}$$

$$n_{5,3} < * 13 \cdot 19 \cdot 3 \cdot 7 \cdot 11 < * n_{6,5} \text{ and } n_{3,5} < * n_{6,5} .$$

$$\sigma^*(n_{6,5})/n_{6,5} < 2^9/257 . \text{ Therefore } 3 \parallel n_j \text{ and } 5 \parallel n_j$$

for the remaining classes j . Using this fact modify the n_j 's and let

$$\begin{aligned}
 n_3 &= 5 \cdot 3 \cdot 7 \cdot 19 \cdot 31 \\
 * > n_5 &= 5 \cdot 3 \cdot 7 \cdot 19 \cdot 11^2 \\
 n_6 &= 5 \cdot 13 \cdot 3 \cdot 7 \cdot 19 \\
 n_{10} &= 5 \cdot 13 \cdot 37 \cdot 3 \cdot 7
 \end{aligned}$$

$\sigma^*(n_3)/n_3 < 2^9/257$. Therefore classes 3 and 5 can be eliminated. $\sigma^*(n_j)/n_j > 2$ for $j = 6$ and 10 by (2.3) .

Step 3.

Classes 6 and 10 remain to be considered.

$$\begin{aligned} n_{6,7} &= 5 \cdot 13 \cdot 3 \cdot 19 \cdot 31 \\ * > n_{10,7} &= 5 \cdot 13 \cdot 37 \cdot 3 \cdot 19 \\ n_{6,13} &= 5 \cdot 37 \cdot 3 \cdot 7 \cdot 19 \\ * > n_{10,13} &= 5 \cdot 37 \cdot 61 \cdot 3 \cdot 7 \end{aligned}$$

$n_{6,7} < * n_{6,13} \cdot \sigma^*(n_{6,13})/n_{6,13} < 2^9/257$. Therefore 3, 5, 7, and 13 are all unitary divisors of n_6 and n_{10} .

Since $7 \parallel n_{10}$, $m = 7$ if n is in class 10.

It is known that there are no unitary perfect numbers with $m = 7$. Therefore, class 10 can be eliminated.

By Lemma 4 part (a.) the other unitary divisor, p^α , of n_6 must be > 64 , and since $3 \parallel n_6$ and $5 \parallel n_6$, it is known that $3 \nmid (1 + p^\alpha)$. Using Table II consider various possible values of p^α .

If $p^\alpha = 67$ then $m = 8$ which implies that

$$257 \mid n_6 \text{ which is impossible.}$$

If $p^\alpha = 79$ then $m = 10$ which implies that

$$5^2 \mid n_6 \text{ which is a contradiction.}$$

Let $p^\alpha = 103$. Then $\sigma^*(n_6)/n_6 < 2^9/257$. Thus, there are no unitary perfect numbers with $r = 5$.

Proof of Theorem 1 part (7). If $m = 6$, then

$$\underline{N = 87360.}$$

Step 1.

$$1 + 2^6 = 5 \cdot 13 \text{ which implies that } a \geq 2.$$

Class	a	b	c	n_j
1	2	1	2	$5 \cdot 13 \cdot 3 \cdot 7 \cdot 11^2$
2	3	0	2	$<* 5 \cdot 13 \cdot 17 \cdot 3 \cdot 7$
3	2	3	1	$5 \cdot 13 \cdot 3 \cdot 7^2 \cdot 11^2 \cdot 19^2$
4	3	2	1	$<* 5 \cdot 13 \cdot 17 \cdot 3 \cdot 7^2 \cdot 11^2$
5	4	1	1	$<* 5 \cdot 13 \cdot 17 \cdot 29 \cdot 3 \cdot 7^2$
6	5	0	1	$<* 5 \cdot 13 \cdot 17 \cdot 29 \cdot 37 \cdot 3$
7	2	5	0	$5 \cdot 13 \cdot 3^2 \cdot 7^2 \cdot 11^2 \cdot 19^2 \cdot 23^2$
8	3	4	0	$<* 5 \cdot 13 \cdot 17 \cdot 3^2 \cdot 7^2 \cdot 11^2 \cdot 19^2$
9	4	3	0	$<* 5 \cdot 13 \cdot 17 \cdot 29 \cdot 3^2 \cdot 7^2 \cdot 11^2$
10	5	2	0	$<* 5 \cdot 13 \cdot 17 \cdot 29 \cdot 37 \cdot 3^2 \cdot 7^2$
11	6	1	0	$<* 5 \cdot 13 \cdot 17 \cdot 29 \cdot 37 \cdot 41 \cdot 3^2$
12	7	0	0	$*> 5 \cdot 13 \cdot 17 \cdot 29 \cdot 37 \cdot 41 \cdot 53$

$$\sigma^*(n_j)/n_j < 2^7/65 \text{ for } j = 6 \text{ and } 11. \quad \sigma^*(n_1)/n_1 > 2^7/65.$$

Therefore all classes except 1 and 2 can be eliminated from consideration.

Step 2.

It has already been proved that there are no unitary perfect numbers with $r = 5$. Therefore only the following subclasses of classes 1 and 2 remain to be considered.

Subclass	a'	b'	c'	k_j
1	1	1	2	
2	2	0	2	$5 \cdot 13 \cdot 3 \cdot 7$
3	2	1	1	
4	3	0	1	

These four subclasses include all possible sets $K(a', b', c')$ such that $a' \leq 2$, $b' \leq 1$, and $c' \leq 2$, not all equal and such that $a' \leq 3$, $b' = 0$, and $c' \leq 2$, not both equal. Subclass 1 was eliminated with class 4, subclass 3 with class 9, and subclass 4, with class 11.

$\sigma^*(k_2)/k_2 = 2^7/65$. Therefore $N = 2^6 \cdot 5 \cdot 13 \cdot 3 \cdot 7 = 87360$ is the only unitary perfect number with $m = 6$.

Proof of Theorem 1 part (8). If $r = 4$, then

$N = 87360$.

Step 1.

If $c = 0$ then $m = 3$. Therefore there are no unitary perfect numbers with $r = 4$ such that $c = 0$.

Class	a	b	c	n_j
1	0	0	4	$3 \cdot 7 \cdot 11 \cdot 19$
2	0	1	3	$* > 3 \cdot 7 \cdot 11 \cdot 19^2$
3	1	0	3	$5 \cdot 3 \cdot 7 \cdot 11$
4	0	2	2	$3 \cdot 7 \cdot 11^2 \cdot 19^2$
5	1	1	2	$< * 5 \cdot 3 \cdot 7 \cdot 11^2$
6	2	0	2	$< * 5 \cdot 13 \cdot 3 \cdot 7$
7	0	3	1	$3 \cdot 7^2 \cdot 11^2 \cdot 19^2$
8	1	2	1	$< * 5 \cdot 3 \cdot 7^2 \cdot 11^2$
9	2	1	1	$< * 5 \cdot 13 \cdot 3 \cdot 7^2$
10	3	0	1	$< * 5 \cdot 13 \cdot 17 \cdot 3$

$\sigma^*(n_j)/n_j < 2^{5/17}$ for $j = 1$ or 10 . $\sigma^*(n_3)/n_3 > 2^9/257$.
 $2^6 n_6 = 87360$, a unitary perfect number, as determined in
the proof of part (7) of Theorem 1. If there are any other
numbers n in classes 4, 5, or 6 that give unitary perfect
numbers $N = 2^m n$, m will be less than 6. However, all
unitary perfect numbers with $m \leq 6$ are known. Therefore
only class 3 is left to be considered.

Step 2.

$$n_{3,3} = 5 \cdot 7 \cdot 11 \cdot 19$$

$$n_{3,5} = 13 \cdot 3 \cdot 7 \cdot 11$$

Assume $3 \parallel n_3$ and $5 \parallel n_3$ and using this fact let

$$n_3' = 5 \cdot 3 \cdot 7 \cdot 19$$

$n_{3,3} <^* n_{3,5} <^* n_3'$. $\sigma(n_3')/n_3' < 2^9/257$. Therefore $3 \parallel n_3$ and $5 \parallel n_3$ but there are no numbers, n , in class 3 such that $N = 2^m n$ is unitary perfect. Therefore, 87360 is the only unitary perfect number with $r = 4$.

Proof of Theorem 1 part (9). It is not possible for $m = 8$.

Step 1.

$1 + 2^8 = 257$ which implies that $a \neq 0$. All unitary perfect numbers with $r \leq 5$ are known and, therefore, class (0,1,4) can be eliminated from consideration.

Class	a	b	c	n_j
1	1	2	3	$257 \cdot 3 \cdot 7 \cdot 11 \cdot 19^2 \cdot 23^2$
2	2	1	3	$<^* 257 \cdot 5 \cdot 3 \cdot 7 \cdot 11 \cdot 19^2$
3	3	0	3	$<^* 257 \cdot 5 \cdot 13 \cdot 3 \cdot 7 \cdot 11$
4	1	4	2	$257 \cdot 3 \cdot 7 \cdot 11^2 \cdot 19^2 \cdot 23^2 \cdot 31^2$
5	2	3	2	$<^* 257 \cdot 5 \cdot 3 \cdot 7 \cdot 11^2 \cdot 19^2 \cdot 23^2$
6	3	2	2	$<^* 257 \cdot 5 \cdot 13 \cdot 3 \cdot 7 \cdot 11^2 \cdot 19^2$
7	4	1	2	$<^* 257 \cdot 5 \cdot 13 \cdot 17 \cdot 3 \cdot 7 \cdot 11^2$
8	5	0	2	$<^* 257 \cdot 5 \cdot 13 \cdot 17 \cdot 29 \cdot 3 \cdot 7$
9	1	6	1	$257 \cdot 3 \cdot 7^2 \cdot 11^2 \cdot 19^2 \cdot 23^2 \cdot 31^2 \cdot 43^2$

Class	a	b	c	n_j
10	2	5	1	$<* 257 \cdot 5 \cdot 3 \cdot 7^2 \cdot 11^2 \cdot 19^2 \cdot 23^2 \cdot 31^2$
11	3	4	1	$<* 257 \cdot 5 \cdot 13 \cdot 3 \cdot 7^2 \cdot 11^2 \cdot 19^2 \cdot 23^2$
12	4	3	1	$<* 257 \cdot 5 \cdot 13 \cdot 17 \cdot 3 \cdot 7^2 \cdot 11^2 \cdot 19^2$
13	5	2	1	$<* 257 \cdot 5 \cdot 13 \cdot 17 \cdot 29 \cdot 3 \cdot 7^2 \cdot 11^2$
14	6	1	1	$<* 257 \cdot 5 \cdot 13 \cdot 17 \cdot 29 \cdot 37 \cdot 3 \cdot 7^2$
15	7	0	1	$<* 257 \cdot 5 \cdot 13 \cdot 17 \cdot 29 \cdot 37 \cdot 41 \cdot 3$
16	1	8	0	$257 \cdot 3^2 \cdot 7^2 \cdot 11^2 \cdot 19^2 \cdot 23^2 \cdot 31^2 \cdot 43^2 \cdot 47^2$
17	2	7	0	$<* 257 \cdot 5 \cdot 3^2 \cdot 7^2 \cdot 11^2 \cdot 19^2 \cdot 23^2 \cdot 31^2 \cdot 43^2$
18	3	6	0	$<* 257 \cdot 5 \cdot 13 \cdot 3^2 \cdot 7^2 \cdot 11^2 \cdot 19^2 \cdot 23^2 \cdot 31^2$
19	4	5	0	$<* 257 \cdot 5 \cdot 13 \cdot 17 \cdot 3^2 \cdot 7^2 \cdot 11^2 \cdot 19^2 \cdot 23^2$
20	5	4	0	$<* 257 \cdot 5 \cdot 13 \cdot 17 \cdot 29 \cdot 3^2 \cdot 7^2 \cdot 11^2 \cdot 19^2$
21	6	3	0	$<* 257 \cdot 5 \cdot 13 \cdot 17 \cdot 29 \cdot 37 \cdot 3^2 \cdot 7^2 \cdot 11^2$
22	7	2	0	$<* 257 \cdot 5 \cdot 13 \cdot 17 \cdot 29 \cdot 37 \cdot 41 \cdot 3^2 \cdot 7^2$
23	8	1	0	$*> 257 \cdot 5 \cdot 13 \cdot 17 \cdot 29 \cdot 37 \cdot 41 \cdot 53 \cdot 3^2$
24	9	0	0	$*> 257 \cdot 5 \cdot 13 \cdot 17 \cdot 29 \cdot 37 \cdot 41 \cdot 53 \cdot 61$

$\sigma^*(n_j)/n_j < 2^9/257$ for $j = 1, 5, 14$, and 22 .

$\sigma^*(n_j)/n_j > 2^9/257$ for $j = 2, 6$, and 15 .

Therefore all classes except $2, 3, 6, 7, 8$, and 15 can be eliminated from consideration.

Step 2.

$$n_{15,3} = 257 \cdot 5 \cdot 13 \cdot 17 \cdot 29 \cdot 37 \cdot 41 \cdot 7$$

$$n_{15,5} = 257 \cdot 13 \cdot 17 \cdot 5^2 \cdot 29 \cdot 37 \cdot 41 \cdot 3$$

$n_{15,3} < * n_{15,5} \cdot \sigma^*(n_{15,5})/n_{15,5} < 2^9/257$. Therefore
 $3 \parallel n_{15}$ and $5 \parallel n_{15}$. Using this fact modify n_{15} and let

$$n_{15} = 257 \cdot 5 \cdot 13 \cdot 37 \cdot 61 \cdot 73 \cdot 97 \cdot 3$$

$\sigma^*(n_{15})/n_{15} < 2^9/257$. Therefore class 15 can be eliminated.

$$\begin{aligned} n_{6,3} &= 5 \cdot 13 \cdot 257 \cdot 7 \cdot 11 \cdot 3^2 \cdot 19^2 \\ < * n_{7,3} &= 5 \cdot 13 \cdot 17 \cdot 257 \cdot 7 \cdot 11 \cdot 3^2 \\ * > n_{8,3} &= 5 \cdot 13 \cdot 17 \cdot 29 \cdot 257 \cdot 7 \cdot 11 \\ n_{6,5} &= 13 \cdot 17 \cdot 257 \cdot 3 \cdot 7 \cdot 11^2 \cdot 19^2 \\ < * n_{7,5} &= 13 \cdot 17 \cdot 5^2 \cdot 257 \cdot 3 \cdot 7 \cdot 11^2 \\ < * n_{8,5} &= 13 \cdot 17 \cdot 5^2 \cdot 29 \cdot 257 \cdot 3 \cdot 7 \end{aligned}$$

$\sigma^*(n_{j,i})/n_{j,i} < 2^9/257$ for $j,i = (7,3)$ and $(8,5)$.

Therefore $3 \parallel n_j$ and $5 \parallel n_j$ for $j = 6, 7$, and 8 .

Using this fact modify n_6 and let

$$n_6 = 5 \cdot 13 \cdot 257^2 \cdot 3 \cdot 7 \cdot 11^2 \cdot 19^2$$

$\sigma^*(n_6)/n_6 < 2^9/257$. Therefore class 6 can be eliminated.

It can be seen by (2.3) that similar values of n_7

and n_8 are > 2 .

$$\begin{aligned} n_{2,3} &= 5 \cdot 257 \cdot 7 \cdot 11 \cdot 19 \cdot 3^2 \\ * > n_{3,3} &= 5 \cdot 13 \cdot 257 \cdot 7 \cdot 11 \cdot 19 \\ n_{2,5} &= 13 \cdot 257 \cdot 3 \cdot 7 \cdot 11 \cdot 19^2 \\ < * n_{3,5} &= 13 \cdot 17 \cdot 257 \cdot 3 \cdot 7 \cdot 11 \end{aligned}$$

Let $n' = 13 \cdot 19 \cdot 257 \cdot 3 \cdot 7 \cdot 11$. Then

$$n_{3,3} <^* n_{2,3} <^* n' <^* n_{3,5} . \quad \sigma^*(n_{3,5})/n_{3,5} < 2^9/257 .$$

Therefore $3 \parallel n_j$ and $5 \parallel n_j$ for $j = 2$ and 3 . Using this fact modify n_j and let

$$\begin{aligned} n_2 &= 5 \cdot 257 \cdot 3 \cdot 7 \cdot 19 \cdot 11^2 \\ <^* n_3 &= 5 \cdot 13 \cdot 257 \cdot 3 \cdot 7 \cdot 19 \end{aligned}$$

$\sigma^*(n_2)/n_2 < 2^9/257$. Therefore class 2 can be eliminated.

$\sigma^*(n_3)/n_3 > 2$ by (2.3) .

Step 3.

Classes 3, 7, and 8 remain to be considered.

Classes 7 and 8:

Consider only those numbers M in the sets $K(4,1,2)$ and $K(5,0,2)$. If $m = 8$ then $7 \nparallel M$. Let

$$\begin{aligned} M_7 &= 5 \cdot 13 \cdot 37 \cdot 257 \cdot 3 \cdot 19 \cdot 7^2 \\ \Rightarrow M_8 &= 5 \cdot 13 \cdot 37 \cdot 61 \cdot 257 \cdot 3 \cdot 19 \end{aligned}$$

$\sigma^*(M_7)/M_7 < 2^9/257$. Therefore there are no unitary perfect numbers $N = 2^8 n$ with n in $K(4,1,2)$ or in $K(5,0,2)$.

Consider the following subclasses.

Subclass	a'	b'	c'	k_j
1	3	1	2	
2	4	0	2	$5 \cdot 13 \cdot 37 \cdot 257 \cdot 3 \cdot 7$
3	4	1	1	
4	5	0	1	

These four subclasses include all possible sets of values (a', b', c') , $a' \leq 4$, $b' \leq 1$, and $c' \leq 2$, not all equal, and $a' \leq 5$, $b' = 0$, and $c' \leq 2$, not both equal. Subclasses 1, 3, and 4 were eliminated with classes 6, 12, and 13 respectively. Consider subclass 2. It has previously been determined that $3 \parallel k_2$ and $5 \parallel k_2$. $19 \nmid k_2$ since $m = 8$. Thus,

$$k_{2,7} = 5 \cdot 13 \cdot 37 \cdot 257 \cdot 3 \cdot 31$$

$$k_{2,13} = 5 \cdot 37 \cdot 61 \cdot 257 \cdot 3 \cdot 7$$

$$k_{2,7} <^* k_{2,13} \quad \sigma^*(k_{2,13})/k_{2,13} < 2^9/257. \quad \text{Therefore}$$

3, 5, 7, and 13 are unitary divisors for all k such

that k is in subclass 2 and $N = 2^8 k$ is unitary perfect.

Let

$$k_2 = 5 \cdot 13 \cdot 257 \cdot p^\alpha \cdot 3 \cdot 7$$

By Lemma 4 part (a.) $p^\alpha > 64$. It is also known that

$3 \nmid (1 + p^\alpha)$ since $3 \parallel k_2$ and $5 \parallel k_2$. Using these facts

and Table I consider possible values for p^α .

If $p^\alpha = 73$ then $37 \nmid k_2$ which is impossible.

If $p^\alpha = 97$ then $7^2 \nmid k_2$ which is impossible.

If $p^\alpha = 109$ then $11 \nmid k_2$ which is impossible.

Let $p^\alpha = 137$. Then $\sigma^*(k_2)/k_2 < 2^9/257$. Therefore classes 7 and 8 can be eliminated.

Class 3:

All unitary perfect numbers with $r < 6$ are known. Therefore only the numbers M in $K(3,0,3)$ must be considered.

Since $m = 8$, $7 \nmid M$, and any prime p , such that $p \equiv 3 \pmod{4}$ and $2^3 \mid (1+p)$ does not divide M .

Using this fact and Table II let

$$M_3 = 5 \cdot 13 \cdot 257 \cdot 3 \cdot 19 \cdot 43$$

$\sigma^*(M_3)/M_3 < 2^9/257$. Therefore class 3 can be eliminated.

Thus, there are no unitary perfect numbers with $m = 8$.

Proof of Theorem 1 part (9). It is not possible for $m = 9$.

Step 1.

$1 + 2^9 = 3^3 \cdot 19$ which implies that $b + c \geq 2$. All unitary perfect numbers with $r \leq 5$ are known. Therefore class $(0,0,5)$ can be eliminated from consideration.

Class	a	b	c	n_j
1	0	2	4	$7 \cdot 11 \cdot 19 \cdot 23 \cdot 3^4 \cdot 31^2$
2	1	1	4	$<* 5 \cdot 7 \cdot 11 \cdot 19 \cdot 23 \cdot 3^4$
3	2	0	4	$<* 5 \cdot 13 \cdot 3^3 \cdot 7 \cdot 11 \cdot 19$
4	0	4	3	$7 \cdot 11 \cdot 19 \cdot 3^4 \cdot 23^2 \cdot 31^2 \cdot 43^2$
5	1	3	3	$<* 5 \cdot 7 \cdot 11 \cdot 19 \cdot 3^4 \cdot 23^2 \cdot 31^2$
6	2	2	3	$<* 5 \cdot 13 \cdot 7 \cdot 11 \cdot 19 \cdot 3^4 \cdot 23^2$
7	3	1	3	$<* 5 \cdot 13 \cdot 17 \cdot 7 \cdot 11 \cdot 19 \cdot 3^4$
8	4	0	3	$*> 5 \cdot 13 \cdot 17 \cdot 29 \cdot 3^3 \cdot 7 \cdot 19$
9	0	6	2	$7 \cdot 11 \cdot 3^4 \cdot 19^2 \cdot 23^2 \cdot 31^2 \cdot 43^2 \cdot 47^2$
10	1	5	2	$<* 5 \cdot 7 \cdot 11 \cdot 3^4 \cdot 19^2 \cdot 23^2 \cdot 31^2 \cdot 43^2$
11	2	4	2	$<* 5 \cdot 13 \cdot 7 \cdot 11 \cdot 3^4 \cdot 19^2 \cdot 23^2 \cdot 31^2$
12	3	3	2	$<* 5 \cdot 13 \cdot 17 \cdot 7 \cdot 11 \cdot 3^4 \cdot 19^2 \cdot 23^2$
13	4	2	2	$<* 5 \cdot 13 \cdot 17 \cdot 29 \cdot 7 \cdot 11 \cdot 3^4 \cdot 19^2$
14	5	1	2	$*> 5 \cdot 13 \cdot 17 \cdot 29 \cdot 37 \cdot 7 \cdot 19 \cdot 3^4$
15	6	0	2	$*> 5 \cdot 13 \cdot 17 \cdot 29 \cdot 37 \cdot 41 \cdot 3^3 \cdot 19$
16	0	8	1	$7 \cdot 3^4 \cdot 11^2 \cdot 19^2 \cdot 23^2 \cdot 31^2 \cdot 43^2 \cdot 47^2 \cdot 59^2$
17	1	7	1	$<* 5 \cdot 7 \cdot 3^4 \cdot 11^2 \cdot 19^2 \cdot 23^2 \cdot 31^2 \cdot 43^2 \cdot 47^2$
18	2	6	1	$<* 5 \cdot 13 \cdot 7 \cdot 3^4 \cdot 11^2 \cdot 19^2 \cdot 23^2 \cdot 31^2 \cdot 43^2$
19	3	5	1	$<* 5 \cdot 13 \cdot 17 \cdot 7 \cdot 3^4 \cdot 11^2 \cdot 19^2 \cdot 23^2 \cdot 31^2$
20	4	4	1	$<* 5 \cdot 13 \cdot 17 \cdot 29 \cdot 7 \cdot 3^4 \cdot 11^2 \cdot 19^2 \cdot 23^2$
21	5	3	1	$<* 5 \cdot 13 \cdot 17 \cdot 29 \cdot 37 \cdot 7 \cdot 3^4 \cdot 11^2 \cdot 19^2$
22	6	2	1	$<* 5 \cdot 13 \cdot 17 \cdot 29 \cdot 37 \cdot 41 \cdot 7 \cdot 3^4 \cdot 19^2$
23	7	1	1	$*> 5 \cdot 13 \cdot 17 \cdot 29 \cdot 37 \cdot 41 \cdot 53 \cdot 19 \cdot 3^4$

Class	a	b	c	n_j
24	0	10	0	$3^4 \cdot 7^2 \cdot 11^2 \cdot 19^2 \cdot 23^2 \cdot 31^2 \cdot 43^2 \cdot 47^2 \cdot 59^2 \cdot 67^2$
25	1	9	0	$<* 5 \cdot 3^4 \cdot 7^2 \cdot 11^2 \cdot 19^2 \cdot 23^2 \cdot 31^2 \cdot 43^2 \cdot 47^2 \cdot 59^2$
26	2	8	0	$<* 5 \cdot 13 \cdot 3^4 \cdot 7^2 \cdot 11^2 \cdot 19^2 \cdot 23^2 \cdot 31^2 \cdot 43^2 \cdot 47^2$
27	3	7	0	$<* 5 \cdot 13 \cdot 17 \cdot 3^4 \cdot 7^2 \cdot 11^2 \cdot 19^2 \cdot 23^2 \cdot 31^2 \cdot 43^2$
28	4	6	0	$<* 5 \cdot 13 \cdot 17 \cdot 29 \cdot 3^4 \cdot 7^2 \cdot 11^2 \cdot 19^2 \cdot 23^2 \cdot 31^2$
29	5	5	0	$<* 5 \cdot 13 \cdot 17 \cdot 29 \cdot 37 \cdot 3^4 \cdot 7^2 \cdot 11^2 \cdot 19^2 \cdot 23^2$
30	6	4	0	$<* 5 \cdot 13 \cdot 17 \cdot 29 \cdot 37 \cdot 41 \cdot 3^4 \cdot 7^2 \cdot 11^2 \cdot 19^2$
31	7	3	0	$<* 5 \cdot 13 \cdot 17 \cdot 29 \cdot 37 \cdot 41 \cdot 53 \cdot 3^4 \cdot 7^2 \cdot 19^2$
32	8	2	0	$*> 5 \cdot 13 \cdot 17 \cdot 29 \cdot 37 \cdot 41 \cdot 53 \cdot 61 \cdot 3^4 \cdot 19^2$

$\sigma^*(n_j)/n_j < 2^{10}/513$ for $j = 3, 7, 13, 22$, and 31 .

Therefore there are no unitary perfect numbers with $m = 9$.

Proof of Theorem 1 part (9). It is not possible for $m = 10$.

Step 1.

$1 + 2^{10} = 5^2 \cdot 41$ which implies that $a \geq 2$.

Class	a	b	c	n_j
1	2	1	4	$5^2 \cdot 41 \cdot 3 \cdot 7 \cdot 11 \cdot 19 \cdot 23^2$
2	3	0	4	$<* 5^2 \cdot 41 \cdot 13 \cdot 3 \cdot 7 \cdot 11 \cdot 19$
3	2	3	3	$5^2 \cdot 41 \cdot 3 \cdot 7 \cdot 11 \cdot 19^2 \cdot 23^2 \cdot 31^2$
4	3	2	3	$<* 5^2 \cdot 41 \cdot 13 \cdot 3 \cdot 7 \cdot 11 \cdot 19^2 \cdot 23^2$
5	4	1	3	$<* 5^2 \cdot 41 \cdot 13 \cdot 17 \cdot 3 \cdot 7 \cdot 11 \cdot 19^2$
6	5	0	3	$<* 5^2 \cdot 41 \cdot 13 \cdot 17 \cdot 29 \cdot 3 \cdot 7 \cdot 11$

Class	a	b	c	n_j
7	2	5	2	$5^2 \cdot 41 \cdot 3 \cdot 7 \cdot 11^2 \cdot 19^2 \cdot 23^2 \cdot 31^2 \cdot 43^2$
8	3	4	2	$<* 5^2 \cdot 41 \cdot 13 \cdot 3 \cdot 7 \cdot 11^2 \cdot 19^2 \cdot 23^2 \cdot 31^2$
9	4	3	2	$<* 5^2 \cdot 41 \cdot 13 \cdot 17 \cdot 3 \cdot 7 \cdot 11^2 \cdot 19^2 \cdot 23^2$
10	5	2	2	$<* 5^2 \cdot 41 \cdot 13 \cdot 17 \cdot 29 \cdot 3 \cdot 7 \cdot 11^2 \cdot 19^2$
11	6	1	2	$<* 5^2 \cdot 41 \cdot 13 \cdot 17 \cdot 29 \cdot 37 \cdot 3 \cdot 7 \cdot 11^2$
12	7	0	2	$<* 5^2 \cdot 41 \cdot 13 \cdot 17 \cdot 29 \cdot 37 \cdot 53 \cdot 3 \cdot 7$
13	2	7	1	$5^2 \cdot 41 \cdot 3 \cdot 7^2 \cdot 11^2 \cdot 19^2 \cdot 23^2 \cdot 31^2 \cdot 43^2 \cdot 47^2$
14	3	6	1	$<* 5^2 \cdot 41 \cdot 13 \cdot 3 \cdot 7^2 \cdot 11^2 \cdot 19^2 \cdot 23^2 \cdot 31^2 \cdot 43^2$
15	4	5	1	$<* 5^2 \cdot 41 \cdot 13 \cdot 17 \cdot 3 \cdot 7^2 \cdot 11^2 \cdot 19^2 \cdot 23^2 \cdot 31^2$
16	5	4	1	$<* 5^2 \cdot 41 \cdot 13 \cdot 17 \cdot 29 \cdot 3 \cdot 7^2 \cdot 11^2 \cdot 19^2 \cdot 23^2$
17	6	3	1	$<* 5^2 \cdot 41 \cdot 13 \cdot 17 \cdot 29 \cdot 37 \cdot 3 \cdot 7^2 \cdot 11^2 \cdot 19^2$
18	7	2	1	$<* 5^2 \cdot 41 \cdot 13 \cdot 17 \cdot 29 \cdot 37 \cdot 53 \cdot 3 \cdot 7^2 \cdot 11^2$
19	8	1	1	$<* 5^2 \cdot 41 \cdot 13 \cdot 17 \cdot 29 \cdot 37 \cdot 53 \cdot 61 \cdot 3 \cdot 7^2$
20	9	0	1	$*> 5^2 \cdot 41 \cdot 13 \cdot 17 \cdot 29 \cdot 37 \cdot 53 \cdot 61 \cdot 73 \cdot 3$
21	2	9	0	$5^2 \cdot 41 \cdot 3^2 \cdot 7^2 \cdot 11^2 \cdot 19^2 \cdot 23^2 \cdot 31^2 \cdot 43^2 \cdot 47^2 \cdot 59^2$
22	3	8	0	$<* 5^2 \cdot 41 \cdot 13 \cdot 3^2 \cdot 7^2 \cdot 11^2 \cdot 19^2 \cdot 23^2 \cdot 31^2 \cdot 43^2 \cdot 47^2$
23	4	7	0	$<* 5^2 \cdot 41 \cdot 13 \cdot 17 \cdot 3^2 \cdot 7^2 \cdot 11^2 \cdot 19^2 \cdot 23^2 \cdot 31^2 \cdot 43^2$
24	5	6	0	$<* 5^2 \cdot 41 \cdot 13 \cdot 17 \cdot 29 \cdot 3^2 \cdot 7^2 \cdot 11^2 \cdot 19^2 \cdot 23^2 \cdot 31^2$
25	6	5	0	$<* 5^2 \cdot 41 \cdot 13 \cdot 17 \cdot 29 \cdot 37 \cdot 3^2 \cdot 7^2 \cdot 11^2 \cdot 19^2 \cdot 23^2$
26	7	4	0	$<* 5^2 \cdot 41 \cdot 13 \cdot 17 \cdot 29 \cdot 37 \cdot 53 \cdot 3^2 \cdot 7^2 \cdot 11^2 \cdot 19^2$
27	8	3	0	$<* 5^2 \cdot 41 \cdot 13 \cdot 17 \cdot 29 \cdot 37 \cdot 53 \cdot 61 \cdot 3^2 \cdot 7^2 \cdot 11^2$
28	9	2	0	$<* 5^2 \cdot 41 \cdot 13 \cdot 17 \cdot 29 \cdot 37 \cdot 53 \cdot 61 \cdot 73 \cdot 3^2 \cdot 7^2$
29	10	1	0	$*> 5^2 \cdot 41 \cdot 13 \cdot 17 \cdot 29 \cdot 37 \cdot 53 \cdot 61 \cdot 73 \cdot 89 \cdot 3^2$
30	11	0	0	$*> 5^2 \cdot 41 \cdot 13 \cdot 17 \cdot 29 \cdot 37 \cdot 53 \cdot 61 \cdot 73 \cdot 89 \cdot 97$

$\sigma^*(n_j)/n_j < 2^{11}/1025$ for $j = 1, 4, 11, 19$, and 28 .

$\sigma^*(n_j)/n_j > 2^{11}/1025$ for $j = 2, 5$, and 12 .

Therefore all classes except 2, 5, 6, and 12 can be eliminated from consideration.

Step 2.

Class 12:

Consider only numbers M in $K(7,0,2)$. Then $7 \nmid M$ if $m = 10$. Using this fact let

$$M_{12} = 5^2 \cdot 41 \cdot 13 \cdot 17 \cdot 29 \cdot 37 \cdot 53 \cdot 3 \cdot 11$$

$\sigma^*(M_{12})/M_{12} < 2^{11}/1025$. Therefore there are no unitary perfect numbers $N = 2^{10} n$ with n in $K(7,0,2)$.

Consider the following subclasses.

Subclass	a'	b'	c'	k_j
1	6	0	2	$5^2 \cdot 41 \cdot 13 \cdot 17 \cdot 29 \cdot 37 \cdot 3 \cdot 7$
2	7	0	1	

These two subclasses include all sets of values (a', b', c') such that $a' \leq 7$, $b' = 0$, and $c' \leq 2$, not both equal.

Subclass 2 was eliminated with class 18. $\sigma^*(k_1)/k_1 < 2^{11}/1025$.

Therefore class 12 can be eliminated.

Classes 5 and 6:

Consider only numbers M in $K(4,1,3)$ or in $K(5,0,3)$.

Then if $m = 10$, $7 \nmid M$. Using this fact let

$$M_5 = 5^2 \cdot 41 \cdot 13 \cdot 17 \cdot 3 \cdot 11 \cdot 19 \cdot 7^2$$

$$M_6 = 5^2 \cdot 41 \cdot 13 \cdot 17 \cdot 29 \cdot 3 \cdot 11 \cdot 19$$

$\sigma^*(M_6)/M_6 < 2^{11}/1025$. Therefore there are no unitary perfect numbers $N = 2^{10}n$ with n in $K(4,1,3)$ or in $K(5,0,3)$.

Consider the subclasses

Subclass	a'	b'	c'	k_j
1	3	1	3	
2	4	0	3	$5^2 \cdot 41 \cdot 13 \cdot 17 \cdot 3 \cdot 7 \cdot 11$
3	4	1	2	
4	5	0	2	

These four subclasses include all possible sets of values (a', b', c') such that $a' \leq 4$, $b' \leq 1$, and $c' \leq 3$, not all equal, and $a' \leq 5$, $b' = 0$, and $c' \leq 3$, not both equal. Subclasses 1, 3, and 4 were eliminated with classes 4 and 10.

$$\sigma^*(k_2)/k_2 < 2^{11}/1025 .$$

$$k_{2,3} = 5^2 \cdot 41 \cdot 13 \cdot 17 \cdot 7 \cdot 11 \cdot 19$$

$$k_{2,11} = 5^2 \cdot 41 \cdot 13 \cdot 17 \cdot 3 \cdot 7 \cdot 19$$

$$k_{2,17} = 5^2 \cdot 41 \cdot 13 \cdot 29 \cdot 3 \cdot 7 \cdot 11$$

$\sigma^*(k_{2,17})/k_{2,17} < 2^{11}/1025$. Therefore 3, 11, and 17 are unitary divisors of k_2 which implies that 3^2 divides the left hand side of equation (2.1) which is a contradiction. Therefore classes 5 and 6 can be eliminated.

Class 2:

Consider only the numbers M in $K(3,0,4)$. Then if $m = 10$, $7 \nmid M$. Using this fact let

$$M_2 = 5^2 \cdot 41 \cdot 13 \cdot 3 \cdot 11 \cdot 19 \cdot 23$$

$\sigma^*(M_2)/M_2 < 2^{11}/1025$. Therefore there are no unitary perfect numbers $N = 2^{10} n$ with n in $K(3,0,4)$.

Consider the following subclasses.

Subclass	a'	b'	c'
1	2	0	4
2	3	0	3

These two subclasses include all sets of values (a', b', c') such that $a' \leq 3$, $b' = 0$, and $c' \leq 4$, not both equal. These two subclasses were eliminated with classes 1 and 4. Therefore, class 2 can be eliminated. Thus, there are no unitary perfect numbers with $m = 10$.

Proof of Theorem 1 part (10). It is not possible
for $r = 6$.

Step 1.

If $c = 0$ then $m = 5$. Therefore there are no unitary
perfect numbers with $r = 6$ such that $c = 0$.

Class	a	b	c	n_j
1	0	0	6	$3 \cdot 7 \cdot 11 \cdot 19 \cdot 23 \cdot 31$
2	0	1	5	$* > 3 \cdot 7 \cdot 11 \cdot 19 \cdot 23 \cdot 31^2$
3	1	0	5	$5 \cdot 3 \cdot 7 \cdot 11 \cdot 19 \cdot 23$
4	0	2	4	$3 \cdot 7 \cdot 11 \cdot 19 \cdot 23^2 \cdot 31^2$
5	1	1	4	$5 \cdot 3 \cdot 7 \cdot 11 \cdot 19 \cdot 23^2$
6	2	0	4	$< * 5 \cdot 13 \cdot 3 \cdot 7 \cdot 11 \cdot 19$
7	0	3	3	$3 \cdot 7 \cdot 11 \cdot 19^2 \cdot 23^2 \cdot 31^2$
8	1	2	3	$5 \cdot 3 \cdot 7 \cdot 11 \cdot 19^2 \cdot 23^2$
9	2	1	3	$< * 5 \cdot 13 \cdot 3 \cdot 7 \cdot 11 \cdot 19^2$
10	3	0	3	$< * 5 \cdot 13 \cdot 17 \cdot 3 \cdot 7 \cdot 11$
11	0	4	2	$3 \cdot 7 \cdot 11^2 \cdot 19^2 \cdot 23^2 \cdot 31^2$
12	1	3	2	$< * 5 \cdot 3 \cdot 7 \cdot 11^2 \cdot 19^2 \cdot 23^2$
13	2	2	2	$< * 5 \cdot 13 \cdot 3 \cdot 7 \cdot 11^2 \cdot 19^2$
14	3	1	2	$< * 5 \cdot 13 \cdot 17 \cdot 3 \cdot 7 \cdot 11^2$
15	4	0	2	$< * 5 \cdot 13 \cdot 17 \cdot 29 \cdot 3 \cdot 7$
16	0	5	1	$3 \cdot 7^2 \cdot 11^2 \cdot 19^2 \cdot 23^2 \cdot 31^2$
17	1	4	1	$< * 5 \cdot 3 \cdot 7^2 \cdot 11^2 \cdot 19^2 \cdot 23^2$

Class	a	b	c	n_j
18	2	3	1	$<* 5 \cdot 13 \cdot 3 \cdot 7^2 \cdot 11^2 \cdot 19^2$
19	3	2	1	$<* 5 \cdot 13 \cdot 17 \cdot 3 \cdot 7^2 \cdot 11^2$
20	4	1	1	$<* 5 \cdot 13 \cdot 17 \cdot 29 \cdot 3 \cdot 7^2$
21	5	0	1	$<* 5 \cdot 13 \cdot 17 \cdot 29 \cdot 37 \cdot 3$

$n_7 <* n_4 <* n_1$. $\sigma^*(n_j)/n_j < 2^{11}/1025$ for $j = 1, 13$, and 21 .

By (2.3) $\sigma^*(n_j)/n_j > 2$ for $j = 3, 5, 8$, and 14 .

Therefore all classes except $3, 5, 6, 8, 9, 10, 14$, and 15 can be eliminated from consideration.

Step 2.

$$\begin{aligned}
 n_{3,3} &= 5 \cdot 3^3 \cdot 7 \cdot 11 \cdot 19 \cdot 23 \\
 <* n_{5,3} &= 5 \cdot 7 \cdot 11 \cdot 19 \cdot 23 \cdot 3^2 \\
 *> n_{6,3} &= 5 \cdot 13 \cdot 7 \cdot 11 \cdot 19 \cdot 23 \\
 n_{8,3} &= 5 \cdot 7 \cdot 11 \cdot 19 \cdot 3^2 \cdot 23^2 \\
 <* n_{9,3} &= 5 \cdot 13 \cdot 7 \cdot 11 \cdot 19 \cdot 3^2 \\
 *> n_{10,3} &= 5 \cdot 13 \cdot 17 \cdot 7 \cdot 11 \cdot 19 \\
 n_{14,3} &= 5 \cdot 13 \cdot 17 \cdot 7 \cdot 11 \cdot 3^2 \\
 *> n_{15,3} &= 5 \cdot 13 \cdot 17 \cdot 29 \cdot 7 \cdot 11 \\
 n_{3,5} &= 13 \cdot 3 \cdot 7 \cdot 11 \cdot 19 \cdot 23 \\
 *> n_{5,5} &= 13 \cdot 3 \cdot 7 \cdot 11 \cdot 19 \cdot 23^2 \\
 n_{6,5} &= 13 \cdot 17 \cdot 3 \cdot 7 \cdot 11 \cdot 19 \\
 *> n_{8,5} &= 13 \cdot 3 \cdot 7 \cdot 11 \cdot 19^2 \cdot 23^2 \\
 n_{9,5} &= 13 \cdot 17 \cdot 3 \cdot 7 \cdot 11 \cdot 19^2 \\
 <* n_{10,5} &= 13 \cdot 17 \cdot 5^2 \cdot 3 \cdot 7 \cdot 11
 \end{aligned}$$

$$\begin{aligned} n_{14,5} &= 13 \cdot 17 \cdot 5^2 \cdot 3 \cdot 7 \cdot 11^2 \\ <* n_{15,5} &= 13 \cdot 17 \cdot 5^2 \cdot 29 \cdot 3 \cdot 7 \end{aligned}$$

$$\begin{aligned} n_{5,3} <* n_{9,3} <* n_{14,3} \cdot n_{3,5} <* n_{6,5} * > n_{10,5} * > n_{15,5} \cdot \\ n_{14,3} <* n_{6,5} \cdot \sigma^*(n_{6,5})/n_{6,5} < 2^{11}/1025 \cdot \text{ Therefore} \\ 3 \parallel n_j \text{ and } 5 \parallel n_j \text{ for } j = 3, 5, 6, 8, 9, 10, 14, \text{ and } 15 \cdot \end{aligned}$$

Classes 14 and 15:

Since $m > 10$ and $3 \parallel n_j$ for $j = 14$ and 15 , a number ≥ 31 is the other unitary divisor, p^α , of n_j such that $p^\alpha \equiv 3 \pmod{4}$, α odd. Let

$$\begin{aligned} n_{14} &= 5 \cdot 13 \cdot 17 \cdot 3 \cdot 31 \cdot 7^2 \\ <* n_{15} &= 5 \cdot 13 \cdot 17 \cdot 29 \cdot 3 \cdot 31 \end{aligned}$$

$\sigma^*(n_{15})/n_{15} < 2^{11}/1025$. Therefore classes 14 and 15 can be eliminated.

Classes 3, 5, 6, 8, 9, and 10:

Using the fact that $3 \parallel n_j$ and $5 \parallel n_j$ for the remaining classes j , let

$$\begin{aligned} n_3 &= 5 \cdot 3 \cdot 7 \cdot 19 \cdot 31 \cdot 43 \\ * > n_5 &= 5 \cdot 3 \cdot 7 \cdot 19 \cdot 31 \cdot 11^2 \\ <* n_6 &= 5 \cdot 13 \cdot 3 \cdot 7 \cdot 19 \cdot 31 \\ n_8 &= 5 \cdot 3 \cdot 7 \cdot 19 \cdot 11^2 \cdot 23^2 \\ n_9 &= 5 \cdot 13 \cdot 3 \cdot 7 \cdot 19 \cdot 11^2 \\ <* n_{10} &= 5 \cdot 13 \cdot 37 \cdot 3 \cdot 7 \cdot 19 \end{aligned}$$

$\sigma^*(n_8)/n_8 < 2^{11}/1025$. Therefore class 8 can be eliminated

$\sigma^*(n_5)/n_5 > 2^{12}/2049$, and by (2.3) $\sigma^*(n_9)/n_9 > 2$.

Step 3.

Classes 3, 5, 6, 9, and 10 remain to be considered.

$$\begin{aligned} n_{3,7} &= 5 \cdot 3 \cdot 19 \cdot 31 \cdot 43 \cdot 67 \\ <* n_{5,7} &= 5 \cdot 3 \cdot 19 \cdot 31 \cdot 43 \cdot 7^2 \\ <* n_{6,7} &= 5 \cdot 13 \cdot 3 \cdot 19 \cdot 31 \cdot 43 \\ *> n_{9,7} &= 5 \cdot 13 \cdot 3 \cdot 19 \cdot 31 \cdot 7^2 \\ <* n_{10,7} &= 5 \cdot 13 \cdot 37 \cdot 3 \cdot 19 \cdot 31 \end{aligned}$$

$$n_{6,7} <* n_{10,7} . \quad \sigma^*(n_{10,7})/n_{10,7} < 2^{11}/1025 .$$

Therefore $7 \parallel n_j$ for the remaining classes j .

Classes 3 and 5:

$$\begin{aligned} n_{3,19} &= 5 \cdot 3 \cdot 7 \cdot 31 \cdot 43 \cdot 67 \\ *> n_{5,19} &= 5 \cdot 3 \cdot 7 \cdot 31 \cdot 43 \cdot 11^2 \\ n_{3,31} &= 5 \cdot 3 \cdot 7 \cdot 19 \cdot 43 \cdot 67 \\ *> n_{5,31} &= 5 \cdot 3 \cdot 7 \cdot 19 \cdot 43 \cdot 11^2 \end{aligned}$$

$$n_{3,19} <* n_{3,31} . \quad \sigma^*(n_{3,31})/n_{3,31} < 2^{12}/2049 .$$

Therefore 19 and 31 are unitary divisors of n_3 and n_5 .

Thus, for class 5, m must equal 13. However,

$$1 + 2^{13} = 8193 \quad \text{and} \quad 3 \nmid 8193 . \quad \text{This is a contradiction since}$$

3 and 5 are both unitary divisors of n_5 . Therefore

class 5 can be eliminated.

3, 5, 7, 19, and 31 are all unitary divisors of n_3 .

Let

$$n_3 = 5 \cdot 3 \cdot 7 \cdot 19 \cdot 31 \cdot p^\alpha$$

In order for $\sigma^*(n_3)/n_3$ to be less than 2 it is necessary for p^α to be greater than 151. If

$$n' = 5 \cdot 3 \cdot 7 \cdot 19 \cdot 31 \cdot 200$$

then $\sigma^*(n')/n' < 2^{11}/1025$. Therefore using Table II the only possibilities for p^α are 163 and 199.

If $p^\alpha = 163$ then $41 | n_3$ which is impossible.

If $p^\alpha = 199$ then $5^2 | n_3$ which is impossible.

Therefore class 3 can be eliminated.

Classes 6, 9, and 10:

$$\begin{aligned} n_{6,13} &= 5 \cdot 37 \cdot 3 \cdot 7 \cdot 19 \cdot 31 \\ * > n_{9,13} &= 5 \cdot 37 \cdot 3 \cdot 7 \cdot 19 \cdot 11^2 \\ < * n_{10,13} &= 5 \cdot 37 \cdot 61 \cdot 3 \cdot 7 \cdot 19 \end{aligned}$$

$n_{10,13} < * n_{6,13}$. $\sigma^*(n_{9,13})/n_{9,13} < 2^{11}/1025$. Therefore $13 \nmid n_9$. $\sigma^*(n_{10,13})/n_{10,13} > 2^{12}/2049$.

3, 5, 7, and 13 are all unitary divisors of n_9

and therefore by Lemma 4 part (a.) all other unitary divisors of n_9 must be > 64 . Since 3 and 5 are both unitary divisors of n_9 there can be no unitary divisor, p^α ,

of n_9 such that $3|(1+p^\alpha)$. Tables I and II are used here and in following parts of the proof to determine possible values for unknown unitary divisors of n_j , $j = 6, 9$, and 10 .

If $67||n_9$ then $17|n_9$ which is impossible.

If $79||n_9$ then $m = 11$ which implies that

$3^2|n_9$, a contradiction.

Noting that $11^2 \nmid n_9$ since $61 \nmid n_9$ let

$$n_9 = 5 \cdot 13 \cdot 3 \cdot 7 \cdot 103 \cdot 19^2$$

$\sigma^*(n_9)/n_9 < 2^{11}/1025$. Therefore class 9 can be eliminated.

Since $m > 10$ and $3||n_{10}$ and $7||n_{10}$ the other unitary divisor p^α , $p \equiv 3 \pmod{4}$, α odd, is ≥ 31 .

Assume $13 \nmid n_{10}$. Let

$$n_{10,13} = 5 \cdot 37 \cdot 61 \cdot 3 \cdot 7 \cdot 31$$

$\sigma^*(n_{10,13})/n_{10,13} < 2^{11}/1025$. Therefore $13||n_{10}$ and all other unitary divisors of n_{10} are > 64 by Lemma 4 part (a).

If $73||n_{10}$ then $37|n_{10}$ which is impossible.

If $97||n_{10}$ then $7^2|n_{10}$, a contradiction.

If $67||n_{10}$ then $m = 9$ which is impossible.

If $79||n_{10}$ then $m = 11$ which implies that

$683|n_{10}$, which is impossible.

If $103||n_{10}$ then $m = 10$ which is impossible.

If $127 \parallel n_{10}$ then $m = 14$ which implies that

$29 \mid n_{10}$ and $113 \mid n_{10}$ which is impossible.

If $139 \parallel n_{10}$ then $m = 9$ which is impossible.

If $151 \parallel n_{10}$ then $m = 10$ which is impossible.

If $163 \parallel n_{10}$ then $m = 9$ which is impossible.

Let

$$n_{10} = 5 \cdot 13 \cdot 109 \cdot 3 \cdot 7 \cdot 191$$

$\sigma^*(n_{10})/n_{10} < 2^{11}/1025$. Therefore class 10 can be eliminated.

Assume $13 \nmid n_6$ and $19 \nmid n_6$ and let

$$n_{6,13,19} = 5 \cdot 37 \cdot 3 \cdot 7 \cdot 31 \cdot 43$$

$\sigma^*(n_{6,13,19})/n_{6,13,19} < 2^{11}/1025$. Therefore $13 \parallel n_6$ or $19 \parallel n_6$. 13 and 19 cannot both be unitary divisors of n_6

because then $\sigma^*(n_6)/n_6 > 2$ by (2.3). Assume $19 \parallel n_6$.

Then $13 \nmid n_6$. Let

$$n_6' = 5 \cdot 37 \cdot 3 \cdot 7 \cdot 19 \cdot p^\alpha$$

If $p^\alpha = 31$ then $m = 13$ which implies that

$32 \mid n_6'$, a contradiction.

If $p^\alpha = 43$ then $m = 10$ which is impossible.

If $p^\alpha = 67$ then $m = 10$ which is impossible.

If $p^\alpha = 79$ then $m = 12$ which implies that

$17 \mid n_6'$ and $241 \mid n_6'$ which is impossible.

Let $p_1^\alpha = 103$. Then $\sigma^*(n_6)/n_6 < 2^{11}/1025$. Therefore $13 \parallel n_6$ and $19 \nmid n_6$. Now let

$$n_6 = 5 \cdot 13 \cdot 3 \cdot 7 \cdot p_1^\alpha p_2^\alpha$$

By Lemma 4 part (a) $p_i^\alpha \geq 64$, $i = 1$ and 2 .

If $p_1^\alpha = 67$ then $17 \mid n_6$ which is impossible.

Let $p_1^\alpha = 79$. In order for $\sigma^*(n_6)/n_6$ to be less than 2 it is necessary for p_2^α to be greater than 341.

If $p_2^\alpha = 367$ then $23 \mid n_6$ which is impossible.

If $p_2^\alpha = 379$ then $m = 13$ and $3^2 \mid n_6$, a contradiction.

Let $p_2^\alpha = 419$. Then $\sigma^*(n_6)/n_6 < 2^{12}/2049$. If $m = 11$ then $3^2 \mid n_6$ which is a contradiction. Therefore $p_1^\alpha \neq 79$.

Thus, according to Table II, the lowest possible value of p_1^α is 103. If $p_1^\alpha = 103$ then in order for $\sigma^*(n_6)/n_6$ to be less than 2 it is necessary for p_2^α to be greater

than 170. Let $p_1^\alpha = 103$ and $p_2^\alpha = 199$. Then

$\sigma^*(n_6)/n_6 < 2^{12}/2049$. As before m cannot equal 11.

Thus, $p_1^\alpha \neq 103$. Let $p_1^\alpha = 127$. If $p_2^\alpha = 139$ then $\sigma^*(n_6)/n_6 < 2^{12}/2049$. Therefore class 6 can be eliminated.

Thus, there are no unitary perfect numbers with $r = 6$.

Proof of Theorem 2.

Step 1.

$1 + 2^{11} = 3 \cdot 683$ which implies that $b + c \geq 2$. All unitary perfect numbers with $r \leq 6$ are known. Therefore class $(0,0,6)$ can be eliminated from consideration.

Class	a	b	c	n_j
1	0	2	5	$3 \cdot 7 \cdot 11 \cdot 19 \cdot 23 \cdot 31^2 \cdot 683^2$
2	1	1	5	$<* 5 \cdot 3 \cdot 7 \cdot 11 \cdot 19 \cdot 23 \cdot 683^2$
3	2	0	5	$<* 5 \cdot 13 \cdot 3 \cdot 7 \cdot 11 \cdot 19 \cdot 683$
4	0	4	4	$3 \cdot 7 \cdot 11 \cdot 19 \cdot 23^2 \cdot 31^2 \cdot 43^2 \cdot 683^2$
5	1	3	4	$<* 5 \cdot 3 \cdot 7 \cdot 11 \cdot 19 \cdot 23^2 \cdot 31^2 \cdot 683^2$
6	2	2	4	$<* 5 \cdot 13 \cdot 3 \cdot 7 \cdot 11 \cdot 19 \cdot 23^2 \cdot 683^2$
7	3	1	4	$<* 5 \cdot 13 \cdot 17 \cdot 3 \cdot 7 \cdot 11 \cdot 19 \cdot 683^2$
8	4	0	4	$*> 5 \cdot 13 \cdot 17 \cdot 29 \cdot 3 \cdot 7 \cdot 11 \cdot 683$
9	0	6	3	$3 \cdot 7 \cdot 11 \cdot 19^2 \cdot 23^2 \cdot 31^2 \cdot 43^2 \cdot 47^2 \cdot 683^2$
10	1	5	3	$<* 5 \cdot 3 \cdot 7 \cdot 11 \cdot 19^2 \cdot 23^2 \cdot 31^2 \cdot 43^2 \cdot 683^2$
11	2	4	3	$<* 5 \cdot 13 \cdot 3 \cdot 7 \cdot 11 \cdot 19^2 \cdot 23^2 \cdot 31^2 \cdot 683^2$
12	3	3	3	$<* 5 \cdot 13 \cdot 17 \cdot 3 \cdot 7 \cdot 11 \cdot 19^2 \cdot 23^2 \cdot 683^2$
13	4	2	3	$<* 5 \cdot 13 \cdot 17 \cdot 29 \cdot 3 \cdot 7 \cdot 11 \cdot 19^2 \cdot 683^2$
14	5	1	3	$<* 5 \cdot 13 \cdot 17 \cdot 29 \cdot 37 \cdot 3 \cdot 7 \cdot 11 \cdot 683^2$
15	6	0	3	$*> 5 \cdot 13 \cdot 17 \cdot 29 \cdot 37 \cdot 41 \cdot 3 \cdot 7 \cdot 683$
16	0	8	2	$3 \cdot 7 \cdot 11^2 \cdot 19^2 \cdot 23^2 \cdot 31^2 \cdot 43^2 \cdot 47^2 \cdot 59^2 \cdot 683^2$
17	1	7	2	$<* 5 \cdot 3 \cdot 7 \cdot 11^2 \cdot 19^2 \cdot 23^2 \cdot 31^2 \cdot 43^2 \cdot 47^2 \cdot 683^2$

Class	a	b	c	n_j
18	2	6	2	$<* 5 \cdot 13 \cdot 3 \cdot 7 \cdot 11^2 \cdot 19^2 \cdot 23^2 \cdot 31^2 \cdot 43^2 \cdot 683^2$
19	3	5	2	$<* 5 \cdot 13 \cdot 17 \cdot 3 \cdot 7 \cdot 11^2 \cdot 19^2 \cdot 23^2 \cdot 31^2 \cdot 683^2$
20	4	4	2	$<* 5 \cdot 13 \cdot 17 \cdot 29 \cdot 3 \cdot 7 \cdot 11^2 \cdot 19^2 \cdot 23^2 \cdot 683^2$
21	5	3	2	$<* 5 \cdot 13 \cdot 17 \cdot 29 \cdot 37 \cdot 3 \cdot 7 \cdot 11^2 \cdot 19^2 \cdot 683^2$
22	6	2	2	$<* 5 \cdot 13 \cdot 17 \cdot 29 \cdot 37 \cdot 41 \cdot 3 \cdot 7 \cdot 11^2 \cdot 683^2$
23	7	1	2	$<* 5 \cdot 13 \cdot 17 \cdot 29 \cdot 37 \cdot 41 \cdot 53 \cdot 3 \cdot 7 \cdot 683^2$
24	8	0	2	$*> 5 \cdot 13 \cdot 17 \cdot 29 \cdot 37 \cdot 41 \cdot 53 \cdot 61 \cdot 3 \cdot 683$
25	0	10	1	$3 \cdot 7^2 \cdot 11^2 \cdot 19^2 \cdot 23^2 \cdot 31^2 \cdot 43^2 \cdot 47^2 \cdot 59^2 \cdot 67^2 \cdot 683^2$
26	1	9	1	$<* 5 \cdot 3 \cdot 7^2 \cdot 11^2 \cdot 19^2 \cdot 23^2 \cdot 31^2 \cdot 43^2 \cdot 47^2 \cdot 59^2 \cdot 683^2$
27	2	8	1	$<* 5 \cdot 13 \cdot 3 \cdot 7^2 \cdot 11^2 \cdot 19^2 \cdot 23^2 \cdot 31^2 \cdot 43^2 \cdot 47^2 \cdot 683^2$
28	3	7	1	$<* 5 \cdot 13 \cdot 17 \cdot 3 \cdot 7^2 \cdot 11^2 \cdot 19^2 \cdot 23^2 \cdot 31^2 \cdot 43^2 \cdot 683^2$
29	4	6	1	$<* 5 \cdot 13 \cdot 17 \cdot 29 \cdot 3 \cdot 7^2 \cdot 11^2 \cdot 19^2 \cdot 23^2 \cdot 31^2 \cdot 683^2$
30	5	5	1	$<* 5 \cdot 13 \cdot 17 \cdot 29 \cdot 37 \cdot 3 \cdot 7^2 \cdot 11^2 \cdot 19^2 \cdot 23^2 \cdot 683^2$
31	6	4	1	$<* 5 \cdot 13 \cdot 17 \cdot 29 \cdot 37 \cdot 41 \cdot 3 \cdot 7^2 \cdot 11^2 \cdot 19^2 \cdot 683^2$
32	7	3	1	$<* 5 \cdot 13 \cdot 17 \cdot 29 \cdot 37 \cdot 41 \cdot 53 \cdot 3 \cdot 7^2 \cdot 11^2 \cdot 683^2$
33	8	2	1	$<* 5 \cdot 13 \cdot 17 \cdot 29 \cdot 37 \cdot 41 \cdot 53 \cdot 61 \cdot 3 \cdot 7^2 \cdot 683^2$
34	9	1	1	$*> 5 \cdot 13 \cdot 17 \cdot 29 \cdot 37 \cdot 41 \cdot 53 \cdot 61 \cdot 73 \cdot 3 \cdot 683^2$
35	0	12	0	$3^2 \cdot 7^2 \cdot 11^2 \cdot 19^2 \cdot 23^2 \cdot 31^2 \cdot 43^2 \cdot 47^2 \cdot 59^2 \cdot 67^2 \cdot 71^2 \cdot 683^2$
36	1	11	0	$<* 5 \cdot 3^2 \cdot 7^2 \cdot 11^2 \cdot 19^2 \cdot 23^2 \cdot 31^2 \cdot 43^2 \cdot 47^2 \cdot 59^2 \cdot 67^2 \cdot 683^2$
37	2	10	0	$<* 5 \cdot 13 \cdot 3^2 \cdot 7^2 \cdot 11^2 \cdot 19^2 \cdot 23^2 \cdot 31^2 \cdot 43^2 \cdot 47^2 \cdot 59^2 \cdot 683^2$
38	3	9	0	$<* 5 \cdot 13 \cdot 17 \cdot 3^2 \cdot 7^2 \cdot 11^2 \cdot 19^2 \cdot 23^2 \cdot 31^2 \cdot 43^2 \cdot 47^2 \cdot 683^2$
39	4	8	0	$<* 5 \cdot 13 \cdot 17 \cdot 29 \cdot 3^2 \cdot 7^2 \cdot 11^2 \cdot 19^2 \cdot 23^2 \cdot 31^2 \cdot 43^2 \cdot 683^2$
40	5	7	0	$<* 5 \cdot 13 \cdot 17 \cdot 29 \cdot 37 \cdot 3^2 \cdot 7^2 \cdot 11^2 \cdot 19^2 \cdot 23^2 \cdot 31^2 \cdot 683^2$

Class	a	b	c	n_j
41	6	6	0	$<* 5 \cdot 13 \cdot 17 \cdot 29 \cdot 37 \cdot 41 \cdot 3^2 \cdot 7^2 \cdot 11^2 \cdot 19^2 \cdot 23^2 \cdot 683^2$
42	7	5	0	$<* 5 \cdot 13 \cdot 17 \cdot 29 \cdot 37 \cdot 41 \cdot 53 \cdot 3^2 \cdot 7^2 \cdot 11^2 \cdot 19^2 \cdot 683^2$
43	8	4	0	$<* 5 \cdot 13 \cdot 17 \cdot 29 \cdot 37 \cdot 41 \cdot 53 \cdot 61 \cdot 3^2 \cdot 7^2 \cdot 11^2 \cdot 683^2$
44	9	3	0	$<* 5 \cdot 13 \cdot 17 \cdot 29 \cdot 37 \cdot 41 \cdot 53 \cdot 61 \cdot 73 \cdot 3^2 \cdot 7^2 \cdot 683^2$
45	10	2	0	$*> 5 \cdot 13 \cdot 17 \cdot 29 \cdot 37 \cdot 41 \cdot 53 \cdot 61 \cdot 73 \cdot 89 \cdot 3^2 \cdot 683^2$

$n_{34} * > n_{30} \cdot \sigma^*(n_j)/n_j < 2^{12}/2049$ for $j = 1, 4, 9, 18, 29$,
 and 44 . $\sigma^*(n_j)/n_j > 2^{12}/2049$ for $j = 2, 5, 8, 10, 15, 19$,
 24, and 30 . Therefore all classes except 2, 3, 5-8,
 10-15, 19-24, and 30-34 can be eliminated from consideration.

Step 2.

$$\begin{aligned}
 n_{30,3} &= 5 \cdot 13 \cdot 17 \cdot 29 \cdot 37 \cdot 7 \cdot 3^2 \cdot 11^2 \cdot 19^2 \cdot 23^2 \cdot 683^2 \\
 <* n_{31,3} &= 5 \cdot 13 \cdot 17 \cdot 29 \cdot 37 \cdot 41 \cdot 7 \cdot 3^2 \cdot 11^2 \cdot 19^2 \cdot 683^2 \\
 <* n_{32,3} &= 5 \cdot 13 \cdot 17 \cdot 29 \cdot 37 \cdot 41 \cdot 53 \cdot 7 \cdot 3^2 \cdot 11^2 \cdot 683^2 \\
 <* n_{33,3} &= 5 \cdot 13 \cdot 17 \cdot 29 \cdot 37 \cdot 41 \cdot 53 \cdot 61 \cdot 7 \cdot 3^2 \cdot 683^2 \\
 *> n_{34,3} &= 5 \cdot 13 \cdot 17 \cdot 29 \cdot 37 \cdot 41 \cdot 53 \cdot 61 \cdot 73 \cdot 683 \cdot 3^2 \\
 n_{30,5} &= 13 \cdot 17 \cdot 5^2 \cdot 29 \cdot 37 \cdot 3 \cdot 7^2 \cdot 11^2 \cdot 19^2 \cdot 23^2 \cdot 683^2 \\
 <* n_{31,5} &= 13 \cdot 17 \cdot 5^2 \cdot 29 \cdot 37 \cdot 41 \cdot 3 \cdot 7^2 \cdot 11^2 \cdot 19^2 \cdot 683^2 \\
 <* n_{32,5} &= 13 \cdot 17 \cdot 5^2 \cdot 29 \cdot 37 \cdot 41 \cdot 53 \cdot 3 \cdot 7^2 \cdot 11^2 \cdot 683^2 \\
 <* n_{33,5} &= 13 \cdot 17 \cdot 5^2 \cdot 29 \cdot 37 \cdot 41 \cdot 53 \cdot 61 \cdot 3 \cdot 7^2 \cdot 683^2 \\
 *> n_{34,5} &= 13 \cdot 17 \cdot 5^2 \cdot 29 \cdot 37 \cdot 41 \cdot 53 \cdot 61 \cdot 73 \cdot 3 \cdot 683^2
 \end{aligned}$$

Note that $5 \cdot 7 \cdot 3^2 * > 5^2 \cdot 3 \cdot 7^2$. Therefore $n_{33,5} < * n_{33,3}$.

$\sigma^*(n_{33,3})/n_{33,3} < 2^{12}/2049$. Therefore 3 and 5 are unitary divisors of n_j for $j = 30-34$ which is a contradiction since $3 \mid (1 + 2^{11})$. Thus classes 30-34 can be eliminated.

$$\begin{aligned}
 n_{19,3} &= 5 \cdot 13 \cdot 17 \cdot 7 \cdot 11 \cdot 3^2 \cdot 19^2 \cdot 23^2 \cdot 31^2 \cdot 683^2 \\
 < * n_{20,3} &= 5 \cdot 13 \cdot 17 \cdot 29 \cdot 7 \cdot 11 \cdot 3^2 \cdot 19^2 \cdot 23^2 \cdot 683^2 \\
 < * n_{21,3} &= 5 \cdot 13 \cdot 17 \cdot 29 \cdot 37 \cdot 7 \cdot 11 \cdot 3^2 \cdot 19^2 \cdot 683^2 \\
 < * n_{22,3} &= 5 \cdot 13 \cdot 17 \cdot 29 \cdot 37 \cdot 41 \cdot 7 \cdot 11 \cdot 3^2 \cdot 683^2 \\
 * > n_{23,3} &= 5 \cdot 13 \cdot 17 \cdot 29 \cdot 37 \cdot 41 \cdot 53 \cdot 7 \cdot 683 \cdot 3^2 \\
 * > n_{24,3} &= 5 \cdot 13 \cdot 17 \cdot 29 \cdot 37 \cdot 41 \cdot 53 \cdot 61 \cdot 3^3 \cdot 683 \\
 n_{19,5} &= 13 \cdot 17 \cdot 5^2 \cdot 3 \cdot 7 \cdot 11^2 \cdot 19^2 \cdot 23^2 \cdot 31^2 \cdot 683^2 \\
 < * n_{20,5} &= 13 \cdot 17 \cdot 5^2 \cdot 29 \cdot 3 \cdot 7 \cdot 11^2 \cdot 19^2 \cdot 23^2 \cdot 683^2 \\
 < * n_{21,5} &= 13 \cdot 17 \cdot 5^2 \cdot 29 \cdot 37 \cdot 3 \cdot 7 \cdot 11^2 \cdot 19^2 \cdot 683^2 \\
 < * n_{22,5} &= 13 \cdot 17 \cdot 5^2 \cdot 29 \cdot 37 \cdot 41 \cdot 3 \cdot 7 \cdot 11^2 \cdot 683^2 \\
 < * n_{23,5} &= 13 \cdot 17 \cdot 5^2 \cdot 29 \cdot 37 \cdot 41 \cdot 53 \cdot 3 \cdot 7 \cdot 683^2 \\
 * > n_{24,5} &= 13 \cdot 17 \cdot 5^2 \cdot 29 \cdot 37 \cdot 41 \cdot 53 \cdot 61 \cdot 3 \cdot 683
 \end{aligned}$$

$\sigma^*(n_{j,i})/n_{j,i} < 2^{12}/2049$ for $j,i = (22,5), (24,5), (20,3),$
 and $(23,3)$. $\sigma^*(n_{j,i})/n_{j,i} > 2^{12}/2049$ for $j,i = (23,5)$
 and $(21,3)$. Therefore 3 is a unitary divisor of n_j for
 $j = 19, 20, 23,$ and $24,$ and 5 is a unitary divisor of n_j
 for $j = 19-22$ and 24 . Thus classes 19, 20, and 24 can be
 eliminated.

$$\begin{aligned}
 n_{10,3} &= 5 \cdot 7 \cdot 11 \cdot 19 \cdot 3^2 \cdot 23^2 \cdot 31^2 \cdot 43^2 \cdot 683^2 \\
 < * n_{11,3} &= 5 \cdot 13 \cdot 7 \cdot 11 \cdot 19 \cdot 3^2 \cdot 23^2 \cdot 31^2 \cdot 683^2
 \end{aligned}$$

$$\begin{aligned}
<* n_{12,3} &= 5 \cdot 13 \cdot 17 \cdot 7 \cdot 11 \cdot 19 \cdot 3^2 \cdot 23^2 \cdot 683^2 \\
<* n_{13,3} &= 5 \cdot 13 \cdot 17 \cdot 29 \cdot 7 \cdot 11 \cdot 19 \cdot 3^2 \cdot 683^2 \\
*> n_{14,3} &= 5 \cdot 13 \cdot 17 \cdot 29 \cdot 37 \cdot 7 \cdot 11 \cdot 683 \cdot 3^2 \\
*> n_{15,3} &= 5 \cdot 13 \cdot 17 \cdot 29 \cdot 37 \cdot 41 \cdot 3^3 \cdot 7 \cdot 683 \\
n_{10,5} &= 13 \cdot 3 \cdot 7 \cdot 11 \cdot 19^2 \cdot 23^2 \cdot 31^2 \cdot 43^2 \cdot 683^2 \\
<* n_{11,5} &= 13 \cdot 17 \cdot 3 \cdot 7 \cdot 11 \cdot 19^2 \cdot 23^2 \cdot 31^2 \cdot 683^2 \\
<* n_{12,5} &= 13 \cdot 17 \cdot 5^2 \cdot 3 \cdot 7 \cdot 11 \cdot 19^2 \cdot 23^2 \cdot 683^2 \\
<* n_{13,5} &= 13 \cdot 17 \cdot 5^2 \cdot 29 \cdot 3 \cdot 7 \cdot 11 \cdot 19^2 \cdot 683^2 \\
<* n_{14,5} &= 13 \cdot 17 \cdot 5^2 \cdot 29 \cdot 37 \cdot 3 \cdot 7 \cdot 11 \cdot 683^2 \\
*> n_{15,5} &= 13 \cdot 17 \cdot 5^2 \cdot 29 \cdot 37 \cdot 41 \cdot 3 \cdot 7 \cdot 683
\end{aligned}$$

$\sigma^*(n_{j,i})/n_{j,i} < 2^{12}/2049$ for $j,i = (12,5), (15,5), (11,3),$

and $(15,3)$. $\sigma^*(n_{j,i})/n_{j,i} > 2^{12}/2049$ for $j,i = (13,5),$

$(12,3)$, and $(14,3)$. Therefore 3 is a unitary divisor of

n_j for $j = 10, 11,$ and $15,$ and 5 is a unitary divisor of

n_j for $j = 10, 11, 12,$ and $15.$ Thus classes 10, 11, and 15

can be eliminated.

$$\begin{aligned}
n_{5,3} &= 5 \cdot 7 \cdot 11 \cdot 19 \cdot 23 \cdot 3^2 \cdot 31^2 \cdot 683^2 \\
<* n_{6,3} &= 5 \cdot 13 \cdot 7 \cdot 11 \cdot 19 \cdot 23 \cdot 3^2 \cdot 683^2 \\
<* n_{7,3} &= 5 \cdot 13 \cdot 17 \cdot 7 \cdot 11 \cdot 19 \cdot 683 \cdot 3^2 \\
*> n_{8,3} &= 5 \cdot 13 \cdot 17 \cdot 29 \cdot 3^3 \cdot 7 \cdot 11 \cdot 683 \\
n_{5,5} &= 13 \cdot 3 \cdot 7 \cdot 11 \cdot 19 \cdot 23^2 \cdot 31^2 \cdot 683^2 \\
<* n_{6,5} &= 13 \cdot 17 \cdot 3 \cdot 7 \cdot 11 \cdot 19 \cdot 23^2 \cdot 683^2 \\
<* n_{7,5} &= 13 \cdot 17 \cdot 5^2 \cdot 3 \cdot 7 \cdot 11 \cdot 19 \cdot 683^2 \\
*> n_{8,5} &= 13 \cdot 17 \cdot 5^2 \cdot 29 \cdot 3 \cdot 7 \cdot 11 \cdot 683
\end{aligned}$$

$$\sigma^*(n_{j,i})/n_{j,i} < 2^{12}/2049 \quad \text{for } j,i = (5,5) \text{ and } (7,3).$$

$$\sigma^*(n_{j,i})/n_{j,i} > 2^{12}/2049 \quad \text{for } j,i = (6,5) \text{ and } (8,5).$$

Therefore 3 is a unitary divisor of n_j for $j = 5, 6, 7$, and 8, and 5 is a unitary divisor of n_5 . Thus class 5 can be eliminated.

$$n_{2,3} = 5 \cdot 7 \cdot 11 \cdot 19 \cdot 23 \cdot 683 \cdot 3^2$$

$$*> n_{3,3} = 5 \cdot 13 \cdot 3^3 \cdot 7 \cdot 11 \cdot 19 \cdot 683$$

$$n_{2,5} = 13 \cdot 3 \cdot 7 \cdot 11 \cdot 19 \cdot 23 \cdot 683^2$$

$$<* n_{3,5} = 13 \cdot 17 \cdot 3 \cdot 7 \cdot 11 \cdot 19 \cdot 683$$

Note that $5 \cdot 23 \cdot 3^2 < * 13 \cdot 17 \cdot 3$. Therefore $n_{2,3} < * n_{3,5}$.

$\sigma^*(n_{3,5})/n_{3,5} < 2^{12}/2049$. Therefore 3 and 5 are unitary divisors of n_2 and n_3 . Thus classes 2 and 3 can be eliminated.

Classes 6-8, 12-14, and 21-23 remain to be considered.

Step 3.

Classes 12, 31, and 22:

It is known that 5 is a unitary divisor of n_j for $j = 12, 21$, and 22. Therefore, since $3 \mid (1 + 2^{11})$, $3^2 \mid n$.

Assume $3^2 \nmid n$ and let

$$n_{12,9} = 5 \cdot 13 \cdot 17 \cdot 7 \cdot 11 \cdot 19 \cdot 3^4 \cdot 23^2 \cdot 683^2$$

$$n_{21,9} = 5 \cdot 13 \cdot 17 \cdot 29 \cdot 37 \cdot 7 \cdot 11 \cdot 3^4 \cdot 19^2 \cdot 683^2$$

$$<* n_{22,9} = 5 \cdot 13 \cdot 17 \cdot 29 \cdot 37 \cdot 41 \cdot 7 \cdot 11 \cdot 3^4 \cdot 683^2$$

$\sigma^*(n_{j,i})/n_{j,i} < 2^{12}/2049$ for $j,i = (12,9)$ and $(22,9)$.

Therefore $3^2 \nmid n_j$. Using this fact modify the n_j 's and let

$$n_{12} = 5 \cdot 13 \cdot 37 \cdot 7 \cdot 19 \cdot 31 \cdot 3^2 \cdot 11^2 \cdot 683^2$$

$$n_{21} = 5 \cdot 13 \cdot 37 \cdot 61 \cdot 73 \cdot 7 \cdot 19 \cdot 3^2 \cdot 11^2 \cdot 683^2$$

$$<* n_{22} = 5 \cdot 13 \cdot 37 \cdot 61 \cdot 73 \cdot 97 \cdot 7 \cdot 19 \cdot 3^2 \cdot 683^2$$

$\sigma^*(n_j)/n_j > 2^{12}/2049$ for $j = 12$ and 22 .

Therefore classes 12, 21, and 22 can be eliminated.

Classes 6, 7, 8, and 23:

It is known that 3 is a unitary divisor of n_j for $j = 6, 7, 8$, and 23 . Therefore there can be no unitary divisor, p^α , of n_j such that $3 \mid (1 + p^\alpha)$. Using this fact modify the n_j 's and let

$$n_6 = 13 \cdot 37 \cdot 3 \cdot 7 \cdot 19 \cdot 31 \cdot 11^2 \cdot 683^2$$

$$<* n_7 = 13 \cdot 37 \cdot 61 \cdot 3 \cdot 7 \cdot 19 \cdot 31 \cdot 683^2$$

$$*> n_8 = 13 \cdot 37 \cdot 61 \cdot 73 \cdot 3 \cdot 7 \cdot 19 \cdot 683^2$$

$$n_{23} = 13 \cdot 37 \cdot 61 \cdot 73 \cdot 97 \cdot 109 \cdot 157 \cdot 3 \cdot 7 \cdot 683^2$$

$\sigma^*(n_j)/n_j < 2^{12}/2049$ for $j = 7$ and 23 . Therefore classes 6, 7, 8, and 23 can be eliminated.

Classes 13 and 14:

If M represents only those numbers in $K(4,2,3)$ and in $K(5,1,3)$, then $7 \nmid M$. Let

$$\begin{aligned}
M_{13} &= 5 \cdot 13 \cdot 17 \cdot 29 \cdot 3 \cdot 11 \cdot 19 \cdot 7^2 \cdot 683^2 \\
<* M_{14} &= 5 \cdot 13 \cdot 17 \cdot 29 \cdot 37 \cdot 3 \cdot 11 \cdot 19 \cdot 683^2 \\
M_{13,3} &= 5 \cdot 13 \cdot 17 \cdot 29 \cdot 11 \cdot 19 \cdot 23 \cdot 3^2 \cdot 683^2 \\
*> M_{14,3} &= 5 \cdot 13 \cdot 17 \cdot 29 \cdot 37 \cdot 11 \cdot 19 \cdot 683 \cdot 3^2 \\
M_{13,5} &= 13 \cdot 17 \cdot 5^2 \cdot 29 \cdot 3 \cdot 11 \cdot 19 \cdot 7^2 \cdot 683^2 \\
<* M_{14,5} &= 13 \cdot 17 \cdot 5^2 \cdot 29 \cdot 37 \cdot 3 \cdot 11 \cdot 19 \cdot 683^2
\end{aligned}$$

Note that $5 \cdot 23 \cdot 3^2 < * 3 \cdot 5^2 \cdot 37$. Therefore $M_{13,3} < * M_{14,5}$.

$\sigma^*(M_{14,5})/M_{14,5} < 2^{12}/2049$. Therefore 3 and 5 are unitary divisors of M_{13} and M_{14} . Thus all numbers in $K(4,2,3)$ and in $K(5,1,3)$ can be eliminated.

Consider the following subclasses.

Subclass	a'	b'	c'	k_j
1	3	2	3	
2	4	1	3	$5 \cdot 13 \cdot 17 \cdot 29 \cdot 3 \cdot 7 \cdot 11 \cdot 683^2$
3	4	2	2	
4	5	0	3	
5	5	1	2	

These five subclasses include all sets of values of a' , b' , and c' such that $a' \leq 4$, $b' \leq 2$, and $c' \leq 3$, not all equal, and such that $a' \leq 5$, $b' \leq 1$, and $c' \leq 3$, not all equal. Subclass 1 was eliminated with class 12, subclass 3 with class 20, subclass 4 with class 15, and subclass 5 with class 21. Only subclass 2 is left to be

considered. Let

$$k_{2,3} = 5 \cdot 13 \cdot 17 \cdot 29 \cdot 7 \cdot 11 \cdot 683 \cdot 3^2$$

$\sigma^*(k_{2,3})/k_{2,3} < 2^{12}/2049$. Therefore 3 is a unitary divisor of k_2 . Using this fact modify k_2 and let

$$k_2 = 13 \cdot 5^2 \cdot 37 \cdot 61 \cdot 3 \cdot 7 \cdot 19 \cdot 683^2$$

$\sigma^*(k_2)/k_2 < 2^{12}/2049$. Therefore classes 13 and 14 can be eliminated. Thus, there are no unitary perfect numbers with $m = 11$.

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